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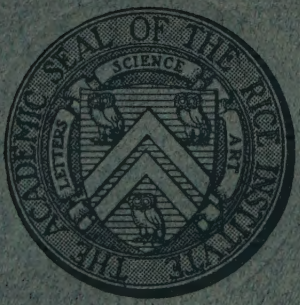

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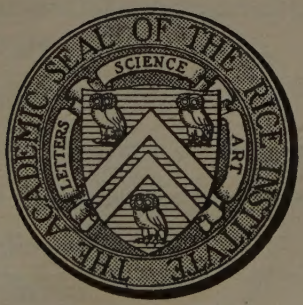

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MODERN RESEARCHES ON THE SINGULARITIES OF FUNCTIONS DEFINED BY TAYLOR'S SERIES[1]

I

PRELIMINARY THEOREMS[2]

1. A *region D*, lying in a plane, is a set of points having the property that each of its points is an interior point, that is, each point of D can be made the center of a circle which contains in its interior only points of D.

Let P, P' be arbitrary points contained in a set E. If, corresponding to an arbitrary positive number ϵ, the points P and P' can be joined by a polygonal line consisting of a finite number of segments, all of its vertices belonging to E,

[1] Lectures delivered at the Rice Institute during the academic year 1926–27 by S. Mandelbrojt, Docteur ès Sciences (Paris), Lecturer in Mathematics at the Rice Institute. Edited by E. R. C. Miles, M.A. (Harvard), Instructor in Mathematics at the Rice Institute.

[2] This first chapter is devoted to definitions and theorems which may be classed as elementary, but in subsequent chapters the author gives more modern and therefore more specialized theorems. The present type of research, which began with the famous thesis of Hadamard, has broadened considerably during the past few years. Complete proofs of all the theorems cannot be compressed in the space available in a short treatise, so the author treats only of such theorems as seem to give unity to the theory, and of the latter theorems proofs are given, for the greater part, in detail.

The reader will find a large bibliography, as well as an enumeration of nearly all the results on the singularities of Taylor's series in Hadamard and Mandelbrojt: *La Série de Taylor et son prolongement analytique*, Scientia, No. 41, also in the author's volume of the Memorial des Sciences Mathématiques. The present treatise may be regarded as complementary to, as well as an elaboration of, some parts of the works just mentioned.

such that the length of each segment is less than ϵ, the set E is said to be *chained.*

A *Cantor continuum* is a set which is closed and chained.

A region is *connected* if two arbitrary points of the region can be joined by a Cantor continuum, consisting entirely of points of the region. Unless otherwise stated, the word "region" will denote a connected region.

If a region D lies entirely in the finite domain, then there are points which are limiting points of D without belonging to D. The aggregate of such points is called the *boundary* of the region D. If we add to the set D the set of boundary points, we obtain a closed set, or, as we shall say, a *closed region.*

2. The equations

$$x = f(t), \qquad y = \phi(t)$$

are said to define a *regular arc* if in the interval $t_0 \leq t \leq t_1$, the following conditions are satisfied:

(*a*) the functions $f(t)$, $\phi(t)$, $f'(t)$ and $\phi'(t)$ are continuous;

(*b*) the equations $f(t) = f(t')$, $\phi(t) = \phi(t')$ admit the unique solution $t = t'$;

(*c*) $f'^2(t) + \phi'^2(t) \neq 0$.

If a finite number of regular arcs are joined consecutively, the resulting curve is said to be regular. A regular curve has therefore a continuously turning tangent except perhaps at a finite number of points.

A set of points which can be placed in one-to-one and continuous correspondence with the totality of points t, $t_0 \leq t \leq t_1$, constitutes a *Jordan curve.* Such a curve may be represented by the equations

$$x = f(t), \qquad y = \phi(t),$$

where $f(t)$ and $\phi(t)$ are continuous in the interval $t_0 \leq t \leq t_1$.

If $(\bar{x}, \bar{y})$ is a point of the set, the pair of equations

$$\bar{x} = f(t), \quad \bar{y} = \phi(t), \quad t_0 < t < t_1,$$

must admit a unique solution $t = \bar{t}$, $t_0 \leq \bar{t} \leq t_1$. This last property is usually indicated by saying that the curve is *simple*. If $f(t_0) = f(t_1)$ and $\phi(t_0) = \phi(t_1)$, the curve is said to be *closed*.

3. The function $f(z)$ is *holomorphic* in a connected region D of the plane $z = x + yi$ if, in each point of D, the following conditions are satisfied:

(a) $f(z)$ is defined;
(b) $f(z)$ is continuous;
(c) $f'(z)$ exists.

The existence of a holomorphic function in a non-connected region is conceivable, as, for example, in a region consisting of the interior and the exterior of a closed curve. We shall see, however, that if the region is composed of n connected subregions, there will exist, in general, n different holomorphic functions.

4. THEOREM 1 (Cauchy-Goursat): *If $f(z)$ is holomorphic in a region D and continuous on the boundary C, consisting of a finite number of closed simple regular curves, then*

$$\int_C f(z)\, dz = 0,$$

the integral being taken along C in the positive sense.

Cauchy's proof of this theorem required the continuity of $f'(z)$. Goursat has given a proof in which only the existence of $f'(z)$ is assumed, as indicated in the statement of the theorem.

THEOREM 2: *If $f(z)$ is holomorphic in D and continuous on the boundary, then, in each interior point of D,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{t - z}. \tag{1}$$

Moreover, the successive derivatives of $f(z)$ may be obtained by differentiating under the integral sign. In fact,

$$f'(z) = \frac{1}{2\pi i}\int_c \frac{f(t)dt,}{(t-z)^2},$$

$$f''(z) = \frac{2!}{2\pi i}\int_c \frac{f(t)dt}{(t-z)^3},$$

$$\cdots\cdot$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_c \frac{f(t)dt}{(t-z)^{n+1}}.$$

Hence *a function holomorphic in a region D has continuous derivatives of all orders in D.*

Consider the series

$$\frac{1}{t-z} = \frac{1}{t-a}\,\frac{1}{1-\dfrac{z-a}{t-a}}$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{(t-a)^{n+1}}, \tag{2}$$

which converges if

$$\left|\frac{z-a}{t-a}\right| < 1. \tag{3}$$

If the point a is within D, then

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{t-z}$$

$$= \frac{1}{2\pi i}\int_c \frac{f(t)dt}{t-a} + \frac{z-a}{2\pi i}\int_c \frac{f(t)dt}{(t-a)^2} + \cdots\cdot \tag{4}$$

For the series (2) converges uniformly for t on C, and z in a circle K, center at a, lying within D, and containing

only points of D in its interior. We may therefore multiply each of its terms by $f(t)dt$ and integrate around C. The resulting series (4) converges for all z in K. We have therefore the following theorem:

THEOREM 3: *The function $f(z)$ can be developed in a Taylor's series about each interior point a of D:*

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots,$$

where

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-a)^{n+1}} = \frac{f^{(n)}(a)}{n!}.$$

In particular, a function holomorphic within a circle can be represented by a Taylor's series which converges uniformly in each closed region lying within the circle. Moreover, a Taylor's series which converges in a circle K, converges uniformly in each closed region interior to K, and represents a function which is holomorphic within K.

5. THEOREM 4: *If the functions $f(z)$, $\phi(z)$ are holomorphic in a region D, and if $f(z) = \phi(z)$ in a subregion D' of D, then $f(z) = \phi(z)$ throughout D.*

The proof may be based on a familiar theorem in the analysis of point sets, namely, the Heine-Borel, or Borel-Lebesgue theorem:

Let E be a closed set lying in the finite domain. If all the points of E are enclosed in a family of open sets O, then the set E is contained in a finite number of the sets O.

Suppose a is a point of D'. Let b be a point of D exterior to D'. Join a and b by a Cantor continuum composed of points of D. Each point of this continuum will be the center of a family of concentric circles. The points of the continuum will be enclosed in a finite number of these circles, all lying within D. The continuum will then lie within the region G included within these circles. Inci-

dentally, this shows that two arbitrary points of a region can be joined by a Jordan curve lying within the region.

Construct within G a region formed by a family of equal circles C_0, $C_1, \cdots C_k$, where C_0 has its center at a, the center of C_{i+1} lies within C_i, and b is within C_k. By hypothesis, $f = \phi$ in C_0. Moreover, $f = \phi$ in C_1. For within a circle K concentric with C_1 and lying within C_0 the functions f and ϕ are equal, and are therefore represented by the same Taylor development about the center of K. But since f and ϕ are holomorphic in C_1, the series converges in C_1. Hence the statement is proved. Proceeding in this way we obtain, finally, $f = \phi$ in C_k, hence in b.

The theorem is equivalent to the statement that if f, ϕ are holomorphic in D_1, D_2, respectively, and if $f = \phi$ in a common subregion, then $f = \phi$ throughout the interior of the region common to D_1 and D_2. We say, then, that in the region $D = D_1 + D_2$ we have a *unique* function; the function f is an analytic continuation in D_1 of the function ϕ defined in D_2, and *vice versa*.

6. Theorem 5: *If the series*

$$\sum_{n=0}^{\infty} a_n(x-a)^n \tag{5}$$

converges for $x = x_0$, *it converges absolutely for all* x *such that* $|x-a| < |x_0-a|$, *and converges uniformly in each closed region interior to the circle with center at* a *and radius* $|x_0-a|$. *If the series diverges for* $x = x_1$, *it diverges for all* x *such that* $|x-a| > |x_1-a|$.

If, therefore, the series (5) converges for at least one $x \neq a$ and diverges for some x, there exists a positive number r such that (5) converges absolutely for $|x-a| < r$, and uniformly in each closed region interior to the circle with center at a, radius r, and diverges for $|x-a| > r$.

As to what happens when $|x - a| = r$, no general statement can be made. The series may converge for all such values of x, or for none; it may converge for some such values, and diverge for others. The number r is called the *radius of convergence*, and the circle with center at a and radius r is called the *circle of convergence*. If the series converges for every value of x, that is, if $r = \infty$, the series is said to define an *integral* function. We shall assume in what follows that the series converges for at least one value of x different from a, that is, $r \neq 0$.

As a consequence of Theorem 4, we can say that if the functions $f(x)$, $\phi(x)$ are expanded into two Taylor's series about the points a, b, respectively, where the circles of convergence C_1, C_2 overlap, and if $f = \phi$ in the region common to C_1 and C_2, we have a unique holomorphic function in the region included within the circles C_1 and C_2. If we have n Taylor's series with overlapping circles of convergence $C_1, C_2, \cdots, C_n$, with $f_k = f_{k+1}$ in the region common to C_k and C_{k+1}, we have a unique holomorphic function in the region included within the circles $C_1, C_2, \cdots, C_n$.

7. Definition 1: Let r, C denote, respectively, the radius and the circle of convergence of the series (5). Suppose the points x and a can be joined by a Jordan curve L, and a sequence of overlapping circles of convergence $C, C_1, \cdots, C_n$ of the corresponding Taylor's series for the functions $f(x), f_1(x), \cdots, f_n(x)$, the circles having their centers on L, x being within C_n, and $f_{k+1} = f_k$ in the common part of C_k and C_{k+1}. Then the function $f(x)$ defined by the series (5) is continued to the point x along the curve L. (Thus, beginning with (5), we have defined a holomorphic function in the extended region, containing the point x, by means of an analytic continuation with respect to the curve L.)

Then the point x is said to be *regular* for the function $f(x)$ defined by (5), relative to the curve L.

Let L' be a second curve joining x and a. Consider a second sequence of Taylor's series for the functions $f(x)$, $\phi_1(x), \cdots, \phi_m(x)$, these functions and their respective circles of convergence having the properties mentioned in the preceding paragraph. Then, in general, $f_n(x) \neq \phi_m(x)$.

DEFINITION 2: Consider all the curves L joining the points x and a, and having the property that if b is an arbitrary point of L different from x, then b is regular for $f(x)$ relative to that part of L from a to b. If x is regular for $f(x)$ relative to every such curve L, then x is said to be *regular for the function* $f(x)$. If, in addition, we have, for a regular point x, and for every pair of these curves, $f_n(x) = \phi_m(x)$, then $f(x)$ is *uniform in the point* x. If $f_n(x) = \phi_m(x)$ for every regular point x, and for every pair of such curves, the function $f(x)$ is said to be *uniform*.

DEFINITION 3: If there exists a curve L' with the property stated in Definition 2, yet such that there is no sequence of overlapping circles of convergence $C, C_1, \cdots, C_n$, centers on L, with $f_{k+1} = f_k$ in the region common to C_{k+1} and C_k, and x in C_n, then the point x is *singular* for $f(x)$, relative to L'. If this condition holds for every curve of the kind described in Definition 2, then x is a *singular point of the function* $f(x)$.

For the purposes of the subsequent investigations, it is convenient to have a more general definition of a singular point.

DEFINITION 4: A point x is *singular for the function* $f(x)$ defined by the series (5) if there exists no curve from a to x such that there is a finite number of circles of convergence $C, C_1, \cdots, C_n$, with centers on the curve, and with $f_{k+1} = f_k$ in the region common to C_{k+1} and C_k, the point x lying within C_n.

We shall say that (r, θ), where r is infinite, represents a single point for all θ, called the point at infinity. Make the transformation $x' = \frac{1}{x}$; the point $x = \infty$ is regular, or singular according as the point $x' = 0$ is regular or singular, respectively, for the transformed function.

DEFINITION 5: If every point of a region D bounded by a closed curve C is a singular point (in the sense of Definition 4) for a uniform function defined by a Taylor's series, then the curve C is called a *cut* for the function defined by the series.

As a consequence of Definition 4, the region D of Definition 5 is always simply connected. Moreover, if every point of a closed curve containing the point a in its interior is singular, so also is every point exterior to the curve.

DEFINITION 6: Let x be a point on the circle of convergence C of (5). Suppose a function $\phi(x)$ is given by a Taylor development which converges in a circle C_1. If x is within C_1, and if $f(x) = \phi(x)$ in the region common to C and C_1, the point x on C is called *regular for the series* (5). If there exists no such circle C_1, such that $f = \phi$ in the common region, then x is *singular for the series* (5).

This definition is equivalent to the previous definition of a regular [singular] point for the function defined by (5), relative to a curve joining a and x, the curve containing only interior points of C.

If the point x, regular [singular] for the function defined by (5), is or is not on C, we shall call x regular [singular] for the series (5) whenever there is no ambiguity, as, for example, when $f(x)$ is uniform.

If the series (5) is such that for every point x of the plane the function $f(x)$ is either regular and uniform in x, or singular in x, then we say that (5) defines an analytic

uniform function in the plane, and the set of regular points is called the *region of regularity*, or the *region of existence* of the function. That this set of points actually constitutes a region will be proved in the next chapter.

At a given point, the function, if not uniform, may be regular for some curves, may have different values for different curves, or may be singular for certain curves. All three cases may occur for a point.

Thus, given a Taylor's series, and the notion of analytic continuation, we can characterize the behavior of the function in every point of the plane, by taking the continuation along every curve from the point a to the point in question. This was the point of view of Méray in France and Weierstrass in Germany. That is, the sequence of coefficients in a Taylor's series about a point a gives the complete determination of a function, regular or singular in each point of the plane, relative to various curves.

For example, a point x_0 on the circumference of C is a *pole of order* m for the series (5) if there exist m numbers $A_1, A_2, \cdots, A_m$ $(A_m \neq 0)$ such that on placing

$$\sum_{i=1}^{m} \frac{A_i}{(x-x_0)^i} = \sum_{n=0}^{\infty} b_n(x-a)^n,$$

$$\sum_{n=0}^{\infty} a_n(x-a)^n - \sum_{n=0}^{\infty} b_n(x-a)^n = \sum_{n=0}^{\infty} (a_n - b_n)(x-a)^n, \qquad (6)$$

we have x_0 as a regular point for the series on the right-hand side of (6).

If the point x_0 is not on the circumference of C, and the function defined by the series (5) is uniform in all the plane, then x_0 is a pole of order m of the function defined by the series (5) if x_0 is regular for the function defined by the series (6).

If x_0 is regular with respect to a curve L and the series (6), then x_0 is said to be a pole of order m relative to the curve L.

Similarly a point x_0 on the circumference of C is an *essential singularity* for the series (5) if there exists an infinite sequence of numbers $A_1, A_2, \cdots$ such that the series

$$\sum_{n=1}^{\infty} A_n z^n \tag{7}$$

is convergent in all the plane, and such that x_0 is regular for the series

$$\sum_{n=0}^{\infty} (a_n - b_n)(x - a)^n,$$

where

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{i=1}^{\infty} \frac{A_i}{(x - x_0)^i}, \tag{8}$$

that is, the convergence of the series (7) implies the convergence of the series on the right-hand side of (8) for all $x \neq x_0$.

We can proceed to the general case for essential singularities as above in the case of poles, by replacing

$$\sum_{i=1}^{m} \frac{A_i}{(x - x_0)^i} \text{ by } \sum_{i=1}^{\infty} \frac{A_i}{(x - x_0)^i},$$

where (8) converges for all $x \neq x_0$.

II

ELEMENTARY THEOREMS ON SINGULARITIES

8. The point of view of Méray-Weierstrass mentioned in the preceding chapter is of the nature of an existence theorem: a unique function is defined by the sequence of coefficients in a Taylor's series about a given point. This says nothing about the actual determination of the function, nor gives any information about its properties. Our purpose is, given the Taylor coefficients, to examine the behavior of the function in the entire plane, particularly with reference to the nature and position of the singularities.

THEOREM 1: *The set of singularities of a uniform analytic function is closed.*

Without loss of generality we may take $f(x)$ as given by $\Sigma a_n x^n$. Let P be a limiting point of the set E of singularities of $f(x)$. Suppose P is not a singular point. By definition, there exists a function $g(x)$ holomorphic in a simply connected region D_1 containing P and the circle of convergence C of $\Sigma a_n x^n$ in its interior, and equal to $f(x)$ in C. Hence we can construct a circle C_1 having its center at P, and lying within D_1, such that $f(x)$ has no singularities within C_1. Consequently P is not a limiting point of E.

For the sense of this theorem it is not necessary to consider the point at infinity as a limiting point of an unbounded set; the theorem remains valid, however, if this interpretation is chosen.

If a series converges for all finite x, the problem of locating the singularities ceases to exist. We shall always assume that the radius of convergence is finite.

THEOREM 2: *There exists at least one singular point on the circle of convergence.*

Suppose every point on C is regular. Then each point on C can be made the center of a circle C_1 in which there exists a function $g(x)$, holomorphic in C_1, and equal to $f(x)$ in the region common to C and C_1. By the Borel-Lebesgue theorem, there is a finite number of such circles, each point on C being interior to at least one of them. Hence we can find an annular region D, containing C in its interior, and in which $f(x)$ is regular. It is then possible to construct a circle C_2, concentric with C and of larger radius, such that the circumference of C_2 lies within D. Then $f(x)$ converges within C_2, which is a contradiction.

Thus the radius r of C is the distance from the origin (or the point a if the function is developed about $x = a$) to at least one singular point. In order to determine r when the sequence of coefficients is given we introduce the concept of the *superior limit.*[1]

9. DEFINITION: Given a bounded sequence of real numbers $\{\alpha_n\}$. We say that A is the *superior limit* of the sequence:

$$A = \overline{\lim_{n\to\infty}}\, \alpha_n$$

if, for an arbitrary $\epsilon > 0$, there exists a number m such that

$$\alpha_n < A + \epsilon, \quad n > m,$$

and if there is an infinity of values of n for which

$$\alpha_n > A - \epsilon.$$

If the sequence is unbounded positively, we define

$$\overline{\lim_{n\to\infty}}\, \alpha_n = +\infty.$$

[1] This notion was introduced by Du Bois Reymond.

THEOREM 3: *The radius of convergence is the value r given by*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = A = \frac{1}{r}.$$

Consider the series $\Sigma a_n x^n$. Assuming that A is finite, not zero, we shall prove that the series converges for

$$|x| < \frac{1}{A(1+\epsilon)}, \tag{1}$$

where $\epsilon > 0$ is arbitrary.

There exists, by definition, a number m such that

$$\sqrt[n]{|a_n|} < A\left(1+\frac{\epsilon}{2}\right), \quad n \geq m.$$

Then, for any x which satisfies (1), we have

$$|a_n x^n| < \left(\frac{1+\frac{\epsilon}{2}}{1+\epsilon}\right)^n = \eta^n, \quad n \geq m.$$

The series

$$\sum_{n=m}^{\infty} |a_n x^n|$$

is therefore dominated by the convergent geometric series

$$\sum_{n=m}^{\infty} \eta^n,$$

and converges.

We show next that the series diverges for

$$|x| > \frac{A}{1-\epsilon}.$$

We have, for infinitely many values of n,

$$\sqrt[n]{|a_n|} > A\left(1-\frac{\epsilon}{2}\right).$$

For these values of n,

$$|a_n x^n| > \left(\frac{1-\frac{\epsilon}{2}}{1-\epsilon}\right)^n = \xi^n > 1.$$

Hence the terms of the series do not approach zero, and the series diverges.

This is known as the Cauchy-Hadamard theorem.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$, the series converges for all x. If $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, the series converges only for $x = 0$. Except when the contrary is stated, it will be assumed hereafter that $r = 1$. For the transformation $z = rx$ replaces a circle of radius r in the z-plane by a circle of radius 1 in the x-plane.

10. Denote by $\Delta_n a_1$ the nth difference of the sequence $\{a_n\}$, viz.,

$$\Delta_n a_1 = a_n - C_n^1 a_{n-1} + \cdots + (-1)^n a_1,$$

where C_n^k is coefficient of the $(k+1)$-th term in the expansion of $(a+b)^n$. We may state here a theorem which is merely a particular case of one to be proved later.

THEOREM 4:[1] *Given the series* $\Sigma a_n x^n$. *Let*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\Delta_n a_1|} = K.$$

Then at least one of the two points on the circle of convergence C with the angle $\cos^{-1}\left(1 - \frac{K^2}{2}\right)$ *is singular, provided the function defined by the series has all its singularities on C and is accordingly regular at* ∞.

[1]Mandelbrojt, *La recherche des points singuliers d'une fonction analytique*, Jour. de Liouville, ser. 9, t. 5 (1926), p. 197.

11. It is in general difficult, if not impossible, to determine directly from the sequence of coefficients the position and character of the singular points. As Hadamard has pointed out, it is necessary to begin by selecting particular cases more or less general.[1] We shall study some cases in which the coefficients are of the types most important with regard to singularities.

THEOREM 5 (Hadamard): *If* $a_n > 0$ *for all* n, *then the point* 1 *is singular.*

Let b be an arbitrary point within C:

$$0 < |b| = \rho < 1,$$

$$b = \rho e^{i\phi}.$$

If the point $e^{i\phi}$ on C is regular, there will exist a circle C_1, center at b, containing $e^{i\phi}$ in its interior, such that the series

$$\sum_{n=0}^{\infty} b_n (x - b)^n$$

defines a function which is equal to

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

within the region common to C and C_1, and which converges in C_1. Hence the radius r_1 of C_1 exceeds $1 - \rho$. But in any case, whether $e^{i\phi}$ is regular or not, we have

$$r_1 \geq 1 - \rho,$$

the lower sign holding if $e^{i\phi}$ is singular. Conversely, if $r_1 = 1 - \rho$, then $e^{i\phi}$ is singular. Thus a necessary and sufficient condition that $e^{i\phi}$ be singular is that

$$r_1 = 1 - \rho.$$

[1] Hadamard-Mandelbrojt, *La série de Taylor*, Scientia, 41, 2d edition, Paris (1926), ch. 2.

Since $e^{i\phi}$ must be singular for at least one value of ϕ, say $\phi = \phi_1$, in the interval $0 \leq \phi < 2\pi$, the above equality is verified for $\phi = \phi_1$. Let b_1 denote the point corresponding to ϕ_1, and let b_n^1 denote the corresponding coefficients. Then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n^1|} > \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} \tag{1}$$

for every sequence of coefficients b_n corresponding to a value of ϕ for which $e^{i\phi}$ is regular. But

$$b_n = \frac{f^{(n)}(b)}{n!},$$

and

$$f^{(m)}(b) = n!\, a_n + (n+1)!\, a_{n+1} b + \cdots + \frac{(n+m)!}{m!} a_{n+m} b^m + \cdots \tag{2}$$

Since the a_i are positive, the absolute value of (2) is a maximum if b is real and positive. Hence (1) is not true. This means that $\phi = 0$ is a singular point.

In a later chapter, we shall investigate the nature and position of the singularities, given either the function itself, or certain properties of the sequence of coefficients.

III

THEOREMS OF HADAMARD AND HURWITZ ON THE COMPOSITION OF SINGULARITIES

12. In order to obtain general theorems relative to the singularities of Taylor's series it is necessary to be able to proceed from theorems in which the hypotheses on the coefficients are comparatively simple (as, for example, in Faber's theorem, Chapter VIII) to more complicated cases by means of "theorems of composition." The problem is this: Given two series whose singularities are known, to determine the singularities of series obtained by combining the two given series in various ways. Two theorems of the kind described are those of Hadamard and Hurwitz.

The theorem of Hadamard [1] is the following:

THEOREM 1: *Given the series*

$$\sum a_n x^n, \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{r}, \tag{1}$$

$$\sum b_n x^n, \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} = \frac{1}{r_1}, \tag{2}$$

representing the uniform functions $f(x)$, $\phi(x)$, *respectively. If* γ *is a singularity of the function defined by*

$$\sum a_n b_n x^n, \tag{3}$$

then there exists a singularity α *of* (1), *and a singularity* β *of* (2), *such that* $\gamma = \alpha\beta$.

[1] Acta Math., t. xxii (1898), p. 55.

13. In order to follow easily the method of proof, suppose first that $f(x)$ has only isolated singularities α_i, and that $\phi(x)$ has one and only one singularity β. For convenience, suppose that the α_i are ordered according to increasing absolute values: $|\alpha_1| \leq |\alpha_2| \leq \cdots$.

With the origin as center, construct a circle C_R of radius R such that no α_i is on the circumference, R being otherwise arbitrary.

Suppose C_R contains $\alpha_1, \alpha_2, \cdots, \alpha_k$. Make α_i, $i \leq k$, the center of a small circle C_i having no point in common with C_j or C_R. Join the circumference of each C_i with C_R by a (Jordan) curve l_i in such a way that l_i does not pass through the origin, and has no point in common with l_j or C_j. Denote by C the boundary of the simply connected region thus obtained.

Construct a curve C_β by multiplying each x of C by β. Then C_β is similar to C. The circles C_i are carried into the circles $C_{i\beta}$ with centers $\alpha_{i\beta}$, $i = 1, 2, \cdots, k$; the curves l_i are carried into curves $l_{i\beta}$. Denote by $C_{R\beta}$ the circle into which C_R is carried.

The function

$$F(z) = \frac{1}{2\pi i} \int_C f(x)\, \phi\left(\frac{z}{x}\right) \frac{dx}{x}$$

is holomorphic within C_β. For if x is on C, $\frac{z}{x} \neq \beta$. Moreover, $f(x)$ is regular on C. Since $\phi\left(\frac{z}{x}\right)$ is continuous in z and x, where z is within C_β, $F(z)$ is continuous. Finally, $\phi\left(\frac{z}{x}\right)$ has a continuous partial derivative with respect to z. Hence $F'(z)$ exists for z within C_β, and $F(z)$ is accordingly holomorphic.

With center at the origin and radius r' construct a small circle C' lying entirely within the region bounded by C. Take z, fixed, interior to C_β and also interior to the circle C'_β:

$$|z| = r'|\beta|.$$

Then the function

$$f(x)\,\phi\left(\frac{z}{x}\right)\frac{1}{x}$$

is holomorphic at all points of C' and all points of the open region bounded by C' and C; in fact for all such points

$$\left|\frac{z}{x}\right| \leq \frac{|z|}{r'} < \beta.$$

Hence

$$\int_C f(x)\,\phi\left(\frac{z}{x}\right)\frac{dx}{x} = \int_{C'} f(x)\,\phi\left(\frac{z}{x}\right)\frac{dx}{x}.$$

Moreover

$$\phi\left(\frac{z}{x}\right) = \sum_{n=0}^{\infty} b_n \frac{z^n}{x^n},$$

where the series is absolutely and uniformly convergent in x for x on C', from the above inequality. Since the same is true of the series for $f(x)$, we can evaluate the right hand member directly:

$$\int_{C'}\left(\sum a_k x^k\right)\left(\sum b_n \frac{z^n}{x^n}\right)\frac{dx}{x} = \int_{C'} \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_k b_n z^n x^{k-n-1}\,dx$$

$$= 2\pi i \sum_{n=1}^{\infty} a_n b_n z^n.$$

That is, $F(z)$ is the function defined by (3). But we have seen that $F(z)$ is holomorphic within C_β. Moreover, the circles C_i can be made arbitrarily small, and C_R arbitrarily large. The same is therefore true for $C_{i\beta}$ and $C_{R\beta}$ respectively. Finally, the curves $l_{i\beta}$ are arbitrary, since the

l_i are arbitrary. Consequently $F(z)$ is regular except perhaps at the points $\alpha_i\beta$. The conclusion of the theorem is therefore valid for this case.

14. For the general case, let C_1, containing but not passing through O, be the boundary of an open simply-connected region D which lies within the region of regularity of $f(x)$. Denote by E the set of singularities of $\phi(x)$. Then E is a closed set. Multiply each point of CD (the set complementary to D) by each point of E. Denote by E the resulting set. Then E is closed. The function

$$F(z) = \int_c f(x)\phi\left(\frac{z}{x}\right)\frac{dx}{x}$$

is regular for z in CE.

Remark. If D is chosen so that it contains in its interior a circle with center at the origin and radius $r' < r$, the set CE contains at least the interior points of the circle with center at the origin and radius $r'r_1$. Since CE is not a null set, and is open, it is a region.

Denote by E_1 the set of singularities of $f(x)$; E_1 is closed. Multiply each point of E by each point of E_1, obtaining a set P. Since E_1 is closed, it is clear that if z is in CP we may choose the region D in such a way that z is in CE.

Hence $f(z)$ is regular in CP, that is, the function

$$F(x) = \sum a_n b_n x^n$$

is regular at every point except perhaps at the points γ of the form $\gamma = \alpha\beta$.

The proof just given includes the first case. The truth of the theorem depends on the extended definition of a singular point.[1]

[1] In this connection, see Montel, *Séries de Polynomes*, Paris (1910), p. 33.

15. THEOREM 2 (Hurwitz [1]): *Given two uniform functions* $f(x)$, $\phi(x)$, *both regular at infinity*:

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{x^{n+1}},$$

$$\phi(x) = \sum_{n=0}^{\infty} \frac{b_n}{x^{n+1}},$$

with radii of convergence r, r_1, *respectively. If* γ *is a singularity of*

$$F(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^{n+1}},$$

where $c_n = a_0 b_n + C_n^1 a_1 b_{n-1} + \cdots + a_n b_0$, *then there exists a singularity* α *of* $f(x)$, *and a singularity* β *of* $\phi(x)$ *such that* $\gamma = \alpha + \beta$.

The proof is analogous to that of Hadamard's theorem. Corresponding to the set E will be the set E' composed of all possible sums of points of E and points of CD, where D is chosen as before. The function

$$F(z) = \frac{1}{2\pi i} \int_c f(x)\ \phi(z - x)dx$$

is regular in CE', hence in CP, where P is the set consisting of all possible sums of points of E and points of E_1.

By analogy, suppose now that $|z| > r' + r$, where $r' > r$. Then

$$F(z) = \frac{1}{2\pi i} \int_{C_{r'}} f(x)\ \phi(z - x)dx, \tag{4}$$

[1] Comptes Rendus, t. 128 (1899), p. 350.

where $C_{r'}$ has its center at the origin. We have now $|z - x| > r_1$. The integral (4) is equivalent to

$$\frac{1}{2\pi i}\int_{C_{r'}} f(x) \sum_{n=0}^{\infty} \frac{b_n}{(z-x)^{n+1}}\, dx$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{C_{r'}} f(x) \frac{b_n}{(z-x)^{n+1}}\, dx$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} a_m \int_{C_{r'}} \frac{dx}{x^{m+1}(z-x)^{n+1}}.$$

Now

$$\frac{1}{2\pi i}\int_{C_{r'}} \frac{dx}{x^{m+1}(z-x)^{n+1}} = \frac{(-1)^n}{n!}\left[\frac{d^n x^{-(m+1)}}{dx^n}\right]_{x=z}$$

$$= \frac{(n+m)!}{n!\,m!}\,\frac{1}{z^{m+n+1}}.$$

Hence

$$F(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \frac{(m+n)!}{m!\,n!}\,\frac{1}{z^{m+n+1}}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_n C_{m+n}^{n} z^{-(m+n+1)} \tag{5}$$

$$= \sum_{n=0}^{\infty} (a_0 b_n + C_n^1 a_1 b_{n-1} + \cdots + a_n b_0)\frac{1}{z^{n+1}},$$

since (5) converges absolutely. But the only singularities of $F(z)$ are of the form $\alpha + \beta$, and the theorem is proved.

Theorems complementary to the foregoing will be treated in Chapter X.

IV

MEROMORPHIC FUNCTIONS [1]

16. If, in a region D, the function $f(x)$ defined by the series $\Sigma a_n x^n$ can be represented as

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ has only poles in the finite plane, and $f_2(x)$ is regular in D, then $f(x)$ is said to be *meromorphic* in D.

A series having only poles as singularities on its circle of convergence is said to be meromorphic on the circle of convergence.

We shall obtain necessary and sufficient conditions, due to Hadamard, that a series be meromorphic on the circle of convergence.

Denote by $D_{n,m}$ the symmetric determinant

$$\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+m} & \cdots & \cdots & a_{n+2m} \end{vmatrix},$$

where the a_i are the coefficients of the series $\Sigma a_n x^n$. If R is the radius of convergence, then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m}|} \leq \frac{1}{R^{m+1}}. \tag{1}$$

The proof of this statement depends on two lemmas which we proceed to establish.

[1] For the theorems in this chapter, see Hadamard, Journal de Liouville, ser. 4, t. 8 (1892), p. 101.

LEMMA 1: *Given k sequences of positive numbers,*

$$\alpha_n^{(1)}, \alpha_n^{(2)}, \cdots, \alpha_n^{(k)}.$$

Then

$$\overline{\lim_{n\to\infty}}\left[\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)}\right] \leq \overline{\lim_{n\to\infty}}\,\alpha_n^{(1)}\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(2)}\cdots\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(k)}.$$

Let $\overline{\lim_{n\to\infty}}\,\alpha_n^{(i)} = \rho_i, \quad i = 1, 2, \cdots, k.$ By definition there exists a number n_0 such that, for $n > n_0$, we have

$$\alpha_n^{(i)} < \rho_i\,(1+\epsilon),$$

$$\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)} < \rho_1\rho_2\cdots\rho_k(1+\epsilon)^k, \qquad i = 1, 2, \cdots, k.$$

For an arbitrary $\eta > 0$, we may choose ϵ so small that $(1+\epsilon)^k < 1+\eta$. Then

$$\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)} < \left[\overline{\lim_{n\to\infty}}\,\alpha_n^{(1)}\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(2)}\cdots\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(k)}\right](1+\eta),$$

and therefore

$$\overline{\lim_{n\to\infty}}\left[\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)}\right] \leq \overline{\lim_{n\to\infty}}\,\alpha_n^{(1)}\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(2)}\cdots\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(k)}.$$

A consequence of this lemma is that, for an arbitrary positive integer m,

$$\overline{\lim_{n\to\infty}}\,\sqrt[n]{|\,a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_{m+1}}\,|} \leq \frac{1}{R^{m+1}},$$

where $\alpha_j = n + i_j$, i_j being a fixed number depending on j, and such that $0 \leq i_j \leq 2m$. The inequality holds for each value of j.

In fact, for a given i,

$$\overline{\lim_{n\to\infty}}\,\sqrt[n]{|\,a_{n+i_j}\,|} = \overline{\lim_{n\to\infty}}\,\sqrt[n+i_j]{|\,a_{n+i_j}\,|},$$

since

$$\overline{\lim_{n\to\infty}}\,|\,a_{n+i_j}\,|^{\frac{1}{n+i_j}} = \overline{\lim_{n\to\infty}}\,\{|\,a_{n+i_j}\,|^{\frac{1}{n}}\}^{\frac{n}{n+i_j}}$$

$$= \overline{\lim_{n\to\infty}}\,|\,a_{n+i_j}\,|^{\frac{1}{n}}.$$

Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{n+i_j}|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{R},$$

so that for a given m,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_{m+1}}|} \leq \frac{1}{R^{m+1}}.$$

LEMMA 2: *Given p Taylor's series, with coefficients*

$$\{A_n^{(1)}\}, \{A_n^{(2)}\}, \cdots, \{A_n^{(p)}\},$$

respectively. If

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(k)}|} = \frac{1}{\rho_k}, \quad k = 1,2,\cdots, p,$$

then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(p)}|} \leq \frac{1}{\rho},$$

where ρ is the smallest of the numbers ρ_k.

The series

$$\sum A_n^{(k)} x^n$$

converges for $|x| < \rho_k$, hence for $|x| < \rho$. The series

$$\sum [A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(p)}] x^n$$

has, therefore, a radius of convergence R which is at least as great as ρ:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(p)}|} = \frac{1}{R} \leq \frac{1}{\rho}.$$

Let us write $D_{n,m}$ in the form

$$D_{n,m} = A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(m+1)!}$$

where each term is of the form

$$a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_{m+1}}.$$

Then, as a consequence of the lemmas, we have, for every m,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(k)}|} \leq \frac{1}{R^{m+1}}, \quad k = 1, 2, \cdots, (m+1)!,$$

and

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m}|} \leq \frac{1}{R^{m+1}}.$$

17. THEOREM 1: *If a series has, on the circle of convergence, p poles and no other singularities, then*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,p}|} < \frac{1}{R^{p+1}}.$$

By hypothesis there exists a polynomial $P_p(x) = 1 + A_1x + \cdots + A_px^p$ such that the series

$$\sum b_nx^n = (1 + A_1x + \cdots + A_px^p) \sum a_nx^n,$$

$$b_{k+p} = a_{k+p} + A_1a_{k+p-1} + \cdots + A_pa_k, \tag{2}$$

has a radius of convergence R_1 which exceeds R. The constant term of $P_p(x)$ has been taken as 1, since by hypothesis the given series is regular at the origin. By means of the relation (2), the determinant $D_{n,p}$ reduces to

$$\begin{vmatrix} a_n & \cdots & a_{n+p-1} & b_{n+p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+p} & \cdots & \cdots & b_{n+2p} \end{vmatrix}.$$

Denote by $\Delta_{n,i}$ the minor of b_{n+i}. Then

$$D_{n,p} = \pm \sum_{i=p}^{2p} (-1)^{i-p} b_{n+i} \Delta_{n,i}.$$

Noting that $\Delta_{n,i}$ is a determinant of order p, we obtain

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\Delta_{n,i}|} \leq \frac{1}{R^p}$$

in the same way as for (1). Moreover, we have, by the choice of $P_p(x)$,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} = \frac{1}{R_1} < \frac{1}{R}.$$

Hence, finally,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,p}|} \leq \frac{1}{R^p}\frac{1}{R_1} < \frac{1}{R^{p+1}}.$$

COROLLARY: *The preceding inequality holds if p is replaced by any $m > p$.*

18. THEOREM 2: *If there exists an integer m such that*

$$i) \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m-1}|} = \frac{1}{R^m},$$

and

$$ii) \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m}|} = \frac{1}{R^m R'} < \frac{1}{R^{m+1}},$$

then, for that m,

$$\lim_{n\to\infty} \sqrt[n]{|D_{n,m-1}|} = \frac{1}{R^m}.$$

We wish to show that given $\epsilon > 0$, no matter how small, there exists a number n_0 such that

$$|D_{n,m-1}| > \left(\frac{1-\epsilon}{R^m}\right)^n \text{ for } n > n_0.$$

Consider the determinant

$$D_{n-1,m} = \begin{vmatrix} a_{n-1} & a_n & \cdots & a_{n+m-1} \\ a_n & a_{n+1} & \cdots & a_{n+m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+m-1} & \cdots & \cdots & a_{n+2m-1} \end{vmatrix}.$$

Among the minors $D_{n+1,m-1}$, $D_{n-1,m-1}$, $D_{n,m-1}$ of this determinant we have the relation

$$D_{n+1,m-1}\, D_{n-1,m-1} - D^2_{n,m-1} = D_{n-1,m}\, D_{n+1,m-2}.^1 \qquad (3)$$

[1] Bôcher: *Introduction to Higher Algebra*, New York (1907), p. 33.

Here the m is of course fixed. We shall first use the identity (3) to show that for n large enough we can obtain an inequality of the form

$$|D_{n-1,m}\,D_{n+1,m-2}| < \left[k\alpha\,\frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2n}, \tag{4}$$

where, by taking ϵ small enough the k will be a positive constant < 1. For this purpose we introduce temporarily quantities ϵ', ϵ'', η', η'', all of which may be made arbitrarily small with n large and ϵ small enough. Indeed, we may take ϵ as small as we like.

In fact, given ϵ', we have from *ii*), by taking n large enough,

$$|D_{n-1,m}| < \left[\frac{1+\epsilon'}{R^m R'}\right]^{n-1},$$

and since $|D_{n+1,m-2}| < [(1+\epsilon'')/R^{m-1}]^{n+1}$ for n large enough, we have

$$|D_{n-1,m}\,D_{n+1,m-2}| < \frac{(1+\epsilon')^{n-1}(1+\epsilon'')^{n+1}}{R'^{n-1}\,R^{2mn-n-1}}$$

$$< \left[\frac{R^{\frac{1}{2}+\frac{1}{2n}}}{R'^{\frac{1}{2}-\frac{1}{2n}}}\,\frac{1+\eta'}{R^m}\right]^{2n},$$

where $1+\eta' > (1+\epsilon')^{\frac{1}{2}-\frac{1}{2n}}(1+\epsilon'')^{\frac{1}{2}+\frac{1}{2n}}$,

$$< \left[\sqrt{\frac{R}{R'}}\,\frac{1+\eta''}{R^m}\right]^{2n}, \quad \text{where } 1+\eta'' > R^{\frac{1}{2n}}R'^{\frac{1}{2n}}(1+\eta'),$$

$$= \left[\sqrt{\frac{R}{R'}}\,\frac{1-\frac{\epsilon}{2}}{R^m}\,\frac{1+\eta''}{1-\frac{\epsilon}{2}}\right]^{2n}$$

$$= \left[(1+\eta)\sqrt{\frac{R}{R'}}\,\frac{1-\frac{\epsilon}{2}}{R^m}\,\frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2n}, \text{where } 1+\eta = \frac{1+\eta''}{1-\epsilon}.$$

By writing then $k = (1+\eta)\sqrt{\frac{R}{R'}}$, $\alpha = \left(1 - \frac{\epsilon}{2}\right)/R^m$ the desired inequality is proved.

From (3) and (4), for all n sufficiently large, we have therefore the inequality

$$|D_{n+1,m-1}D_{n-1,m-1} - D^2_{n,m-1}| < \left[k\alpha\frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2n}. \tag{5}$$

Given ϵ, no matter how small, there will be infinitely many n for which $|D_{n,m-1}| > \alpha^n$, by i). If this is true for all n, n sufficiently large, the theorem is proved; otherwise there will be an ϵ, and it may be taken as small as we please, for which there will be a $D_{n_0,m-1}$ exceeding α^{n_0} in absolute value, preceded by a $D_{n_0-1,m-1}$ which in absolute value is at most as great as α^{n_0-1}, and this for n_0 arbitrarily large.

But from (5), taking $n = n_0$, we have

$$\left|\frac{D_{n_0+1,m-1}D_{n_0-1,m-1}}{D^2_{n_0,m-1}}\right| = \left|\frac{D_{n_0-1,m}D_{n_0+1,m-2}}{D^2_{n_0,m-1}}\right|$$

$$< \left[k\frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2n_0} < k^{2n_0}.$$

Hence we have

$$\left|\frac{D_{n_0+1,m-1}D_{n_0-1,m-1}}{D^2_{n_0,m-1}}\right| > 1 - k^{2n_0}$$

Since, however, $|D_{n_0,m-1}| > \alpha^{n_0}$, $|D_{n_0-1,m-1}| \leq \alpha^{n_0-1}$ this inequality gives the following:

$$|D_{n_0+1,m-1}| > \alpha^{n_0+1}(1-k)^{2n_0}, \tag{6}$$

and for the same reason also the following:

$$\left|\frac{D_{n_0+1,m-1}}{D_{n_0,m-1}}\right| > \alpha(1-k^{2n_0}). \tag{7}$$

In particular, for n_0 sufficiently large, k being < 1, we may write (6) in the form

$$\sqrt[n_0+1]{|D_{n_0+1,\,m-1}|} > \alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}. \tag{6'}$$

We wish to obtain inequalities of the form (6), (7), (6'), holding for $n_0 + i$, where i is arbitrary, provided n_0 is sufficiently large. These inequalities are the following:

$$\left|\frac{D_{n_0+i,\,m-1}}{D_{n_0+i-1,\,m-1}}\right| > \alpha\,[1-k^{2n_0}]\,[1-k^{2(n_0+1)}]\cdots[1-k^{2(n_0+i-1)}], \tag{8}$$

$$|D_{n_0+i,\,m-1}| > \alpha^{n_0+i}[1-k^{2n_0}]^i\,[1-k^{2(n_0+1)}]^{i-1}\cdots [1-k^{2(n_0+i-1)}], \tag{9}$$

$$\sqrt[n_0+i]{|D_{n_0+i,\,m-1}|} > \alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}, \tag{10}$$

and we shall prove them by mathematical induction. They hold for $i = 1$; if they hold for i, we shall prove that they hold for $i + 1$.

If we apply (10), in the form

$$|D_{n_0+i,\,m-1}| > \left[\alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{n_0+i},$$

to (5), we have

$$\left|\frac{D_{n_0+i+1,\,m-1}D_{n_0+i-1,\,m-1}}{D^2_{n_0+i,\,m-1}} - 1\right| < \frac{\left[k\alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2(n_0+i)}}{\left[\alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2(n_0+i)}}$$

$$= k^{2(n_0+i)},$$

Consequently

$$\left|\frac{D_{n_0+i+1,\,m-1}D_{n_0+i-1,\,m-1}}{D^2_{n_0+i,\,m-1}}\right| > 1 - k^{2(n_0+i)},$$

and

$$\left|\frac{D_{n_0+i+1,\,m-1}}{D_{n_0+i,\,m-1}}\right| > \left|\frac{D_{n_0+i,\,m-1}}{D_{n_0+i-1,\,m-1}}\right| [1 - h^{2(n_0+i)}].$$

But this applied to (8) yields immediately the corresponding inequality for $i + 1$, namely,

$$\left|\frac{D_{n_0+i+1,\,m-1}}{D_{n_0+i,\,m-1}}\right| > \alpha\,[1 - k^{2n_0}]\cdots[1 - k^{2(n_0+i)}], \tag{11}$$

and if we multiply this inequality by (9), we obtain the second inequality for $(i + 1)$:

$$|D_{n_0+i+1,\,m-1}| > \alpha^{n_0+i+1}\,[1 - k^{2n_0}]^{i+1}\cdots[1 - k^{2(n_0+i)}]. \tag{12}$$

The inequality (10) for $i + 1$ is an immediate consequence of (11) and (12). In fact, from (12),

$$\begin{aligned}
&\frac{1}{\alpha}\sqrt[n_0+i+1]{|D_{n_0+i+1,\,m-1}|}\\
&> [1 - k^{2n_0}]^{\frac{i+1}{n_0+i+1}}\,[1 - k^{2(n_0+1)}]^{\frac{i}{n_0+i+1}}\cdots[1 - k^{2(n_0+i)}]^{\frac{1}{n_0+i+1}}\\
&> [1 - k^{2n_0}]\,[1 - k^{2(n_0+1)}]\cdots[1 - k^{2(n_0+i)}]\\
&> 1 - [k^{2n_0} + k^{2(n_0+1)} + \cdots + k^{2(n_0+i)}]\\
&> 1 - \frac{k^{2n_0}}{1 - k^2}.
\end{aligned}$$

If then, finally, we take n_0 great enough so that

$$\frac{k^{2n_0}}{1 - k^2} < \frac{\epsilon}{2 - \epsilon}$$

the inequality (10) will be established as a consequence of (8) and (9), whatever the value of i.

The inequality (10) is the one sought. In fact, replacing α by its value, $\alpha = \left(1 - \frac{\epsilon}{2}\right)/R^m$, we have

$$|D_{n,m-1}| > \left[\frac{1-\epsilon}{R^m}\right]^n, \quad n > n_0,$$

which is what we wished to prove.

It is natural to say that if α is the unique limit of the sequence $\{\alpha_n\}$:

$$\lim_{n\to\infty} \alpha_n = \alpha,$$

the sequence converges *regularly* to α. This terminology is due to Hadamard.

19. THEOREM 3: *Under the hypotheses of Theorem 2, the series has on the circle of convergence exactly m poles and no other singularities.*

In the first place, suppose that the series, having only poles on the circle of convergence, has $m - r$ poles, r being a positive integer. Then, by the corollary to Theorem 1,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,(m-r)+(r-1)}|} < \frac{1}{R^{(m-r)+(r-1)+1}},$$

that is,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m-1}|} < \frac{1}{R^m},$$

which contradicts hypothesis *i*).

We seek next to determine a polynomial of degree m,

$$P_m(x) = 1 + \sum_{i=1}^{m} A_i x^i, \quad A_m \neq 0,$$

such that the series representing the function

$$\phi(x) = P_m(x)f(x) = \sum b_n x^n,$$

where, except for a finite number of terms

$$b_{n+m} = a_{n+m} + a_{n+m-1}A_1 + \cdots + a_n A_m,$$

converges in a circle of radius greater than R.

We have, by Theorem 2,

$$\lim_{n\to\infty} \sqrt[n]{|D_{n,m-1}|} = \frac{1}{R^m},$$

so that $D_{n,m-1}$ is not zero for $n \geq n_0$. Consequently there exist sets of numbers $A_1^{(n)}, A_2^{(n)}, \cdots, A_m^{(n)}$ which satisfy systems of equations of the form

$$\left.\begin{aligned} &a_{n+m} + a_{n+m-1}A_1^{(n)} + a_{n+m-2}A_2^{(n)} + \cdots + a_nA_m^{(n)} = 0,\\ &a_{n+m+1} + a_{n+m}A_1^{(n)} + \cdots + a_{n+1}A_m^{(n)} = 0,\\ &\qquad\qquad \cdots\cdots\\ &a_{n+2m-1} + a_{n+2m-2}A_1^{(n)} + \cdots + a_{n+m-1}A_m^{(n)} = 0. \end{aligned}\right\} \quad (14)$$

By means of the substitution

$$\delta_h^{(n)} = A_h^{(n+1)} - A_h^{(n)}, \quad h = 1, 2, \cdots, m, \quad n = n_o, n_{o+1}, \cdots,$$

the system of equations becomes

$$\left.\begin{aligned} &a_{n+m}\delta_1^{(n)} + a_{n+m-1}\delta_2^{(n)} + \cdots + a_{n+1}\delta_m^{(n)} = 0,\\ &a_{n+m+1}\delta_1^{(n)} + \cdots + a_{n+2}\delta_m^{(n)} = 0,\\ &\qquad\qquad \cdots\cdots\\ &a_{n+2m-2}\delta_1^{(n)} + \cdots + a_{n+m-1}\delta_m^{(n)} = 0,\\ &a_{n+2m-1}\delta_1^{(n)} + \cdots + a_{n+m}\delta_m^{(n)} = -H_{n+m}, \end{aligned}\right\} \quad (15)$$

where

$$-H_{n+m} + a_{n+2m} + a_{n+2m-1}A_1^{(n)} + \cdots + a_{n+m}A_m^{(n)} = 0. \quad (16)$$

Eliminating $A_1^{(n)}, \cdots, A_m^{(n)}$ from (14) and (16), we obtain

$$H_{n+m} = \frac{D_{n,m}}{D_{n,m-1}},$$

and from (15),

$$\delta_h^{(n)} = -\frac{D_{n+1,m-2}^{(h)}}{D_{n+1,m-1}} H_{n+m}$$

$$= -\frac{D_{n,m}\, D_{n+1,m-2}^{(h)}}{D_{n,m-1}\, D_{n+1,m-1}},$$

where $D^{(h)}_{n+1,m-2}$ is a determinant with $m-1$ rows, composed of the coefficients of the given series. It follows that

$$\left| D^{(h)}_{n+1,m-2} \right| < \left(\frac{1+\epsilon}{R^{m-1}}\right)^n$$

for n sufficiently large. Then, since

$$\lim_{n\to\infty} \sqrt[n]{|D_{n,m-1}|} = \frac{1}{R^m},$$

and

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m}|} = \frac{1}{R^m R'} < \frac{1}{R^{m+1}},$$

there must exist an integer n', independent of h, such that for $n \geq n'$ we shall have

$$\left| D_{n,m} \right| < \left(\frac{1+\epsilon}{R^m R'}\right)^n,$$

$$\left| D_{n,m-1} \right| > \left(\frac{1-\epsilon}{R^m}\right)^n,$$

$$\left| D_{n+1,m-1} \right| > \left(\frac{1-\epsilon}{R^m}\right)^n,$$

$$\left| D^{(h)}_{n+1,m-2} \right| < \left(\frac{1+\epsilon}{R^{m-1}}\right)^n,$$

Accordingly, for $n \geq n'$,

$$|\delta^{(n)}_h| < \left[\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \frac{R}{R'}\right]^n = \left[(1+\eta)\frac{R}{R'}\right]^n,$$

and we shall suppose that ϵ is so small that the quantity in brackets is < 1. Then the series $\sum_{n=n_0}^{\infty} \delta^{(n)}_h$ converges.

Let $A_h = \lim_{n\to\infty} A^{(n)}_h$, $h=1, 2, \cdots, m$. This limit exists, since

$$\sum_{j=n_0}^{n_0+k-1} \delta^{(j)}_h = A^{(n_0+k)}_h - A^{(n_0)}_h$$

We now show that the series $\Sigma b_n x^n$ for $\phi(x)$ converges in a circle of radius greater than R. We have

$$|A_h - A_h^{(n_0+k)}| = \left| \sum_{j=n_0+k}^{\infty} \delta_h^{(j)} \right|$$

$$\leq \sum_{j=n_0+k}^{\infty} \left[(1+\eta) \frac{R}{R'} \right]^j,$$

and

$$b_{n+m} = a_{n+m-1}(A_1 - A_1^{(n)}) + a_{n+m-2}(A_2 - A_2^{(n)}) + \cdots + a_n (A_m - A_m^{(n)}),$$

since $b_{n+m} - a_{n+m} = a_{n+m-1}A_1 + \cdots + a_n A_m$,

and $a_{n+m} = -a_{n+m-1}A_1^{(n)} - \cdots - a_n A_m^{(n)}$.

The inequality shows that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_h - A_h^{(n)}|} \leq \frac{R}{R'}(1+\eta).$$

Consequently

$$\overline{\lim_{n\to\infty}} \sqrt[n+m]{|b_{n+m}|} \leq \frac{R}{R'}(1+\eta)\frac{1}{R} < \frac{1}{R'},$$

and the proof is complete.

V

LACUNARY SERIES [1]

20. Consider an increasing sequence of positive integers $\{\lambda_n\}$. Denote by $\{\lambda'_n\}$ the increasing sequence consisting of the positive integers not contained in $\{\lambda_n\}$. Both sequences are assumed to be infinite. The zero coefficients in the series

$$\sum a_{\lambda_n} x^{\lambda_n} = \sum c_m x^m, \tag{1}$$

where

$$c_m = \begin{cases} a_{\lambda_n} \text{ when } m = \lambda_n, \\ 0 \quad \text{“} \quad m = \lambda'_n, \end{cases}$$

are called *lacunae*, and a series of the form (1) is a *lacunary series*. Weierstrass first, and later Fredholm, gave examples of such series, in which the circle of convergence is a cut.

It is natural to inquire whether the presence of an infinity of lacunae, distributed according to a definite law, is a characteristic property which may supply information relative to the singularities of the function represented by the series.

The number and, to some extent, the types of singular points of a function are invariant with respect to differentiation and integration. Any property of the coefficients from which conclusions may be reached relative to singularities should therefore be of an invariant character with respect to these operations.

The simplest of such properties is given by the lacunae and their distribution (to within a translation).

[1] Mandelbrojt: Ann. de l'École Norm. Sup., t. 20 (1923), p. 413.

The following general theorem concerning lacunary series is due to Hadamard.[1]

THEOREM 1: *If, for the series* (1), *we have*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1 + \delta, \tag{3}$$

where $\delta > 0$ *is arbitrary, the circle of convergence is a cut.*

Fabry has shown that the conclusion holds if (3) is replaced by

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \infty, \tag{4}$$

a condition which is implied by (3). These theorems will be considered in Chapter XII. In this chapter we shall study lacunary series from a different point of view.

21. THEOREM 2: *If*

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = \infty \tag{5}$$

the series (1) *has on the circle of convergence at least one singularity which is not a pole.*

LEMMA: *If*

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) > K, \tag{6}$$

where K *is a positive integer, and if the series* (1) *has only poles on the circle of convergence, the number of poles is at least* $K + 1$.

The requirement (6) is equivalent to the statement that we can extract a partial sequence $\{\lambda_{n_i}\}$, $i = 1, 2, \cdots$, such that

$$\lambda_{n_i+1} - \lambda_{n_i} > K, \quad i = 1, 2, \cdots.$$

Suppose the lemma false, *i.e.*, we assume that the series, having only poles on the circle of convergence, has

[1] *loc. cit.*, p. 116.

$K' < K + 1$ poles. Then, from Theorem 1, Chapter IV, we have

$$\overline{\lim_{r\to\infty}} \sqrt[r]{|D_{r,K'}|} < \frac{1}{R^{K'+1}}, \tag{7}$$

where

$$D_{r,K'} = \begin{vmatrix} c_r & c_{r+1} & \cdots & c_{r+K'} \\ \cdots & \cdots & \cdots & \cdots \\ c_{r+K'} & \cdots & \cdots & c_{r+2K'} \end{vmatrix}$$

Hence

$$\overline{\lim_{r\to\infty}} \sqrt[r]{|D_{r,0}|} = \overline{\lim_{r\to\infty}} \sqrt[r]{|c_r|} = \frac{1}{R}. \tag{8}$$

Let K'' denote the largest integer such that, for $p = 0, 1, 2, \cdots, K''$, we have

$$\overline{\lim_{r\to\infty}} \sqrt[r]{|D_{r,p}|} = \frac{1}{R^{p+1}}.$$

The existence of K'' follows from (7) and (8); moreover, $0 \leq K'' < K'$. And since

$$\overline{\lim_{r\to\infty}} \sqrt[r]{|D_{r,K''+1}|} < \frac{1}{R^{K''+2}},$$

we have, by Theorem 2, Chapter IV,

$$\lim_{r\to\infty} \sqrt[r]{|D_{r,K''}|} = \frac{1}{R^{K''+1}}, \quad K'' + 1 \leq K',$$

which is a contradiction. For in the determinant

$$D_{\lambda_{n_i}+1,K''} = \begin{vmatrix} c_{\lambda_{n_i}+1} & \cdots & c_{\lambda_{n_i}+K''+1} \\ \cdots & \cdots & \cdots \\ c_{\lambda_{n_i}+K''+1} & \cdots & c_{\lambda_{n_i}+1+2K''} \end{vmatrix},$$

all the elements of the first row belong to the sequence $\{\lambda'_n\}$, since

$$\lambda_{n_i+1} - \lambda_{n_i} > K > K'' + 1.$$

Hence $D_{\lambda_{n_i}+1,K''} = 0$ for all i.

We may now proceed with the proof of Theorem 2. If a function has only poles as singularities on the circle of convergence, there must be a finite number of them. Since there is a subsequence $\{\lambda_{n_i}\}$ such that

$$\lim_{i\to\infty} (\lambda_{n_i+1} - \lambda_{n_i}) = \infty,$$

it follows that, for i sufficiently large,

$$\lambda_{n_i+1} - \lambda_{n_i} > K, \text{ arbitrary.}$$

As we have just seen, the function must therefore have, for each K, at least $K+1$ poles, and consequently the poles are infinite in number. This being impossible, there is at least one singularity other than a pole on the circle of convergence.

22. Theorem 3: *It is possible to construct a series* $\Sigma a_{\lambda_n} x^{\lambda_n}$, *for which*

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = \infty, \tag{5}$$

and which has only one singularity in the entire plane.

In order to prove this, we state without proof two theorems on which the demonstration is based.

Theorem 4 (Leau): *Given an integral function* $g(z)$ *of order* $\leq \delta < 1$, i.e., $|g(z)| < e^{r^{\delta+\epsilon}}$ *for* $r > r_\epsilon$, $r = |z|$; *then the function*

$$f(x) = \Sigma g(n) x^n$$

has the point 1 *as its only singularity, and is accordingly regular at infinity.*[1]

Theorem 5: *Given a sequence* $\{\alpha_n\}$ *such that* $\sum \frac{1}{|\alpha_n|^{\delta+\epsilon}}$ *converges. It is possible to construct an integral function of order* $\leq \delta$, *having the* α_n *as zeros, and no others.*[2]

[1] See Chapter VIII.

[2] Borel: *Fonctions entières*, 2d ed. (1921), p. 56.

Proof of Theorem 3. Let s be chosen so that $\frac{1}{2} < s < 1$. Consider the sequence of series

$$\sum_{n=0}^{\infty} \frac{1}{(n^2+m)^s}, \quad m = 0, 1, 2, \cdots \tag{9}$$

Each of these series, being dominated by the series $\sum \frac{1}{n^{2s}}$, is convergent.

Let

$$\sum_{n=0}^{\infty} \epsilon_n = I$$

be a convergent series of positive terms. For each m, choose n_m such that

$$\sum_{n=n_m}^{\infty} \frac{1}{(n^2+m)^s} < \epsilon_m.$$

Then

$$\sum_{m=0}^{\infty} \left[\sum_{n=n_m}^{\infty} \frac{1}{(n^2+m)^s} \right] < I.$$

Let $u_{n,m} = n^2 + m$, $n \geq n_m$. The series

$$\sum_{m=0}^{\infty} \sum_{n=n_m}^{\infty} \frac{1}{u_{n,m}^s}$$

is absolutely convergent. Sum this series by taking the terms in the order of decreasing magnitude: $\frac{1}{U_1}, \frac{1}{U_2}, \cdots$ (In case two or more terms are equal, we remove all except one of them.) Then the series

$$\sum_{k=1}^{\infty} \frac{1}{U_k^s}$$

converges. By Theorem 5, we can construct an integral function $g(z)$, of order $\leq s$, having the U_k as zeros, and no

others. Hence, by Theorem 4, the series

$$\sum_{n=0}^{\infty} g(n)x^n$$

has the point 1 as its only singularity. We write

$$\sum_{n=0}^{\infty} g(n)x^n = \sum_{n=0}^{\infty} b_{\lambda_n} x^{\lambda_n}, \tag{10}$$

and show that the sequence $\{\lambda_n\}$ has the property (5).

Let i be an arbitrary integer. Corresponding to each i choose an integer p_i such that

$$p_i > n_0, n_1, \cdots, n_i.$$

The zeros U_k, being of the form $n^2 + m$, $n \geq n_m$, are given by

$$p_i^2, p_i^2 + 1, \cdots, p_i^2 + i.$$

For if $m \leq i$, n takes on the values $n_m, n_m + 1, \cdots$, thus including $n = p_i$. Hence, for $i = 0, 1, 2, \cdots$, we have

$$\left.\begin{array}{ll} g(p_0^2) = 0 & \\ g(p_1^2) = 0, & g(p_1^2 + 1) = 0 \\ \cdots & \cdots \\ g(p_i^2) = 0, & g(p_i^2 + 1) = 0, \cdots, g(p_i^2 + i) = 0 \end{array}\right\} \tag{11}$$

The integers in (11) are the λ'_n of the series (10). Select a partial sequence $\{\lambda_{n_i}\}$ as follows: Let $\lambda_{n_1} = p_0^2 - 1$, $\lambda_{n_2} = p_1^2 - 1$, $\lambda_{n_2+1} = p_1^2 + 2$, and so on; let λ_{n_i} be the last subscript before p_i^2 which is not a zero of $g(n)$ and let λ_{n_i+1} be the first subscript after $p_i^2 + i$ which is not a zero of $g(n)$.

With the λ_{n_i} so chosen, we have

$$\lim_{i \to \infty} (\lambda_{n_i+1} - \lambda_{n_i}) = \infty.$$

In the case just considered, since the singularity is not a pole, and is unique, it cannot be a branch point, and must be an essential singularity.

Faber [1] has given an example of a series for which

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n}{n} = \infty,$$

and the function defined by the series has only the point 1 on the circle of convergence as a singularity, but has the lemniscate

$$|x(x-1)| = 2$$

as a cut.

The theorem just proved shows that properties (4) and (5) are entirely distinct, since a consequence of (4) is that the circle of convergence is a cut. We also note that the condition (5) gives no information relative either to the number or the position of the singularities of a function. In fact, if k is an arbitrary positive integer, we can form a series $\Sigma a_n x^n$ having exactly k singularities, arbitrarily situated, in the entire plane. For if we let

$$a_n = g(n)\left(\frac{1}{\beta_1^n} + \frac{1}{\beta_2^n} + \cdots + \frac{1}{\beta_k^n}\right),$$

where the $g(n)$ is as before, and the β_i are arbitrary, but distinct, then the function defined by

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c_{\lambda_n} x^{\lambda_n}$$

has $\beta_1, \beta_2, \cdots, \beta_k$ as essential singularities. On the other hand, the sequence $\{\lambda_n\}$ satisfies (5), since $\Sigma a_n x^n$ is the sum of k series, all of which have the same lacunae.

As Montel [2] has remarked, the author's theorems show that the "magnitude" of the lacunae give information

[1] Sitzungsberichte de l'Ac. de Bavière, t. 36 (1906).

[2] Comptes Rendus, 27 mai, 1925.

about the nature of the singularities, whereas their arithmetical distribution affects the number of singularities. By the *magnitude* of a lacuna of subscript n_i is meant the integer k such that

$$a_{\lambda_{n_i}+1} = 0,\ a_{\lambda_{n_i}+2} = 0, \cdots,\ a_{\lambda_{n_i}+k} = 0,$$

$$a_{\lambda_{n_i}} \neq 0,\ a_{\lambda_{n_i}+k+1} \neq 0.$$

23. The following theorem is a generalization of one of the author's theorems. It is a theorem of Ostrowski [1]; a part of the proof is due to Tchebotareff.

THEOREM 6: *If the series* (1) *has the property that*

$$\overline{\lim_{n\to\infty}}\ (\lambda_{n+1} - 2^p\lambda_n) = \infty, \tag{12}$$

where p is a positive integer, or zero, then the function defined by the series cannot be represented as

$$\frac{\phi(x)}{[P(x)]^{\frac{1}{p+1}}},$$

where P is a polynomial, and the radius of convergence of the series for $\phi(x)$ exceeds that of (1).

If $p = 0$, the theorem states that

$$f(x) \neq \frac{\phi(x)}{P(x)},$$

which is Theorem 2; the function $f(x)$ cannot have poles exclusively on its circle of convergence.

By hypothesis there exists a partial sequence $\{\lambda_{n_i}\}$ for which

$$\lim_{i\to\infty}\ (\lambda_{n_i+1} - 2^p\lambda_{n_i}) = \infty. \tag{13}$$

[1] Jahresbuch der deutschen Math.-Verein., Bd. 35 (1926), 9–21 Heft, p. 269.

Suppose the theorem false. If $P(x)$ has zeros elsewhere as well as on the circle of convergence of (1) we can write

$$f(x) = \frac{\phi(x)}{[P(x)]^{\frac{1}{p+1}}} = \frac{\phi_1(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

where the radius of convergence of the series for $\phi_1(x)$ exceeds that of (1), and where the roots of $P_1(x)$ are situated only on the circle of convergence of (1).

Case 1. We assume that there is at least one zero, α, of P for which $\phi_1(\alpha) \neq 0$.

LEMMA: *Given*

$$f_1(x) = \sum_{n=0}^{\infty} a'_{\lambda_n} x^{\lambda_n} = \sum_{n=0}^{\infty} c_n x^n,$$

$$f_2(x) = \sum_{n=0}^{\infty} b_{\lambda_n} x^{\lambda_n} = \sum_{n=0}^{\infty} d_n x^n,$$

both series satisfying the requirement (13). *Then*

$$f_1(x) f_2(x) = \sum_{n=0}^{\infty} l_{\mu_n} x^{\mu_n} = \sum_{n=0}^{\infty} k_n x^n,$$

where

$$\overline{\lim_{n\to\infty}} (\mu_{n+1} - 2^{p-1}\mu_n) = \infty.$$

As before, let $\{\lambda'_n\}$ be the sequence complementary to $\{\lambda_n\}$. Then $c_{\lambda'_n} = d_{\lambda'_n} = 0$. Let $\{m_n\}$ denote an increasing sequence of positive integers. From (13), we have, for each m_j,

$$\lambda_{n_i+1} > 2^p \lambda_{n_i} + m_j, \quad i > i_j.$$

Then every integer l such that $\lambda_{n_i} < l \leq 2^p \lambda_{n_i} + m_j$ belongs to $\{\lambda'_n\}$. Hence

$$c_{\lambda_{n_i}+1} = 0, c_{\lambda_{n_i}+2} = 0, \cdots, c_{2\lambda_{n_i}} = 0, c_{2\lambda_{n_i}+1} = 0, c_{2^p\lambda_{n_i}+m_j} = 0, \quad (14)$$

with a similar set of equations for the d's.

Now
$$k_n = c_n d_0 + c_{n-1} d_1 + \cdots + c_0 d_n,$$
which may be written
$$k_n = c_n d_0 + c_{n-1} d_1 + \cdots + c_{\frac{n}{2}} d_{\frac{n}{2}} + \cdots + c_0 d_n \qquad (15)$$
when n is even, and
$$k_n = c_n d_0 + c_{n-1} d_1 + \cdots + c_{\left[\frac{n}{2}\right]} d_{\left(\frac{n}{2}\right)} + c_{\left(\frac{n}{2}\right)} d_{\left[\frac{n}{2}\right]} + \cdots + c_0 d_n \qquad (16)$$
when n is odd. The symbol $\left(\frac{n}{2}\right)$ denotes the largest integer less than $\frac{n}{2}$, and $\left[\frac{n}{2}\right]$ the smallest integer greater than $\frac{n}{2}$.

From (15) and (16), it follows that if we place, consecutively,
$$\left.\begin{aligned} n &= 2\lambda_{n_i} + 1, \\ &\cdots\cdots \\ n &= 2^p \lambda_{n_i} + m_j, \end{aligned}\right\} i > i_j, \qquad (17)$$
all the corresponding k_n vanish. For in (15) all the c_q preceding and including $c_{\frac{n}{2}}$ vanish, by (14), and in the remaining terms the d_r vanish. Similarly for (16). Hence all the integers
$$2\lambda_{n_i} + 1,\ 2\lambda_{n_i} + 2, \cdots,\ 2^p \lambda_{n_i} + m_j$$
belong to the sequence $\{\mu'_n\}$ complementary to $\{\mu_n\}$.

If we let
$$2\lambda_{n_i} = \mu_{n_i},\ 2^p \lambda_{n_i} + m_j + 1 = \mu_{n_i+1},$$
we have
$$2^p \lambda_{n_i} = 2^{p-1} \mu_{n_i},\ 2^{p-1} \mu_{n_i} + m_j + 1 = \mu_{n_i+1},$$
and consequently
$$\mu_{n_i+1} - 2^{p-1} \mu_{n_i} > m_j.$$

Since this inequality holds for all i, we have the conclusion stated in the lemma.

Returning to the proof of the theorem, we have, from the lemma,

$$[f(x)]^2 = \left[\sum a_{\lambda_n} x^{\lambda_n}\right]\left[\sum a_{\lambda_n} x^{\lambda_n}\right] = \sum a''_{\mu_n} x^{\mu_n},$$

where it is possible to find a partial sequence $\{\mu_{n_i}\}$ such that

$$\lim_{i\to\infty} (\mu_{n_i+1} - 2^{p-1}\mu_{n_i}) = \infty.$$

Denote by v'_n the sequences

$$\left.\begin{array}{l} 2\lambda_{n_i} + 1 \\ \cdots\cdots \\ 2^p\lambda_{n_i} + m_j \end{array}\right\} i > i_j,\ j = 1, 2, \cdots,$$

and by v''_n the complementary sequences. Then, by (14) and (17), we may write

$$f(x) = \sum a'_{v_n} x^{v_n},\ [f(x)]^2 = \sum a''_{v_n} x^{v_n}.$$

Now we can find a partial sequence $\{v_{n_i}\}$ such that

$$\lim_{i\to\infty} (v_{n_i+1} - v_{n_i}) = \infty.$$

If we replace p by $p - 1$ in the lemma, we obtain

$$[f(x)]^3 = f(x)[f(x)]^2 = \left[\sum a'_{v_n} x^{v_n}\right][a''_{v_n} x^{v_n}]$$
$$= \sum a'''_{K_n} x^{K_n}$$

where we can form a partial sequence $\{K_{n_i}\}$ such that

$$\lim_{i\to\infty} (K_{n_i+1} - K_{n_i}) = \infty.$$

Proceeding in this way we obtain, finally,

$$[f(x)]^{p+1} = \sum a^{(p+1)}_{\pi_n} x^{\pi_n},$$

where there exists a partial sequence $\{\pi_{n_i}\}$ for which

$$\lim_{i\to\infty} (\pi_{n_i+1} - \pi_{n_i}) = \infty.$$

We have consequently arrived at a contradiction. For we have assumed that

$$[f(x)]^{p+1} = \frac{[\phi_1(x)]^{p+1}}{P_1(x)} = \Phi(x),$$

and since $\phi_1(\alpha) \neq 0$, the function $\Phi(x)$ has the point α as a pole on the circle of convergence. Moreover, $\Phi(x)$ has only poles as singularities. This, however, we have just seen to be impossible, and the theorem is proved.

Case 2. To show that $f(x)$ cannot be represented in the form

$$\frac{P_1(x)\ \phi_2(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

where the radius of convergence of the series for $\phi_2(x)$ exceeds that for $f(x)$. We now admit multiple zeros of $P_1(x)$, but require that there exist at least one *simple* zero α of $P_1(x)$ which is not a zero of $\phi_2(x)$.

Unless the theorem is true, we have

$$f(x) = [P_1(x)]^r \phi_2(x), \qquad r = \frac{p}{p+1},$$

$$f'(x) = r\,[P_1(x)^{r-1} P_1'(x)\,\phi_2(x) + [P_1(x)]^r \phi_2'(x)$$

$$= \frac{rP_1'(x)\,\phi_2(x) + P_1(x)\,\phi_2'(x)}{[P_1(x)]^{1-r}}$$

$$= \frac{\phi_3(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

where $\phi_3(x)$ has the same properties as the function $\phi_1(x)$ in Case 1. For, by hypothesis, α is not a zero of $P_1'(x)\,\phi_2(x)$, hence not of $\phi_3(x)$.

On the other hand, the derivative series has lacunae which are characterized by (12).

Case 3. The function $f(x)$ cannot be written in the form

$$\frac{(x-\alpha_i)^{k'_i}\phi_2(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

i.e., assuming that $\phi_1(x) = (x-\alpha_i)^{k'_i}\phi_2(x)$, where $\phi_2(\alpha_i) \neq 0$, k'_i is an integer, and α_i is a zero of $P_1(x)$ of order $k_i > k'_i$.

Let $P_1(x) = A(x-\alpha_1)^{k_1}\cdots(x-\alpha_i)^{k_i}\cdots(x-\alpha_i)^{k_r}$, and suppose

$$f(x) = \frac{(x-\alpha_i)^{k'_i}\phi_2(x)}{(x-\alpha_i)^{\frac{k_i}{p+1}}[P_2(x)]^{\frac{1}{p+1}}}$$

$$= (x-\alpha_i)^r \frac{\phi_2(x)}{[P_2(x)]^{\frac{1}{p+1}}}, \quad r = \frac{k_i'(p+1)-k_i}{p+1}.$$

The number r may be taken as positive; otherwise we have Case 1. Suppose for the present that $r < 1$. Then

$$f'(x) = (x-\alpha_i)^{-(1-r)}\frac{\phi_2(x)}{[P_2(x)]^{\frac{1}{p+1}}} + (x-\alpha_i)^r \frac{d}{dx}\frac{\phi_2(x)}{[P_2(x)]^{\frac{1}{p+1}}}$$

$$= \frac{\phi_4(x)}{(x-\alpha_i)^{1-r}[P_2(x)]^{\frac{1}{p+1}}},$$

which reverts to Case 1 by virtue of the statement at the end of Case 2. If $r > 1$, we need only consider $f^{(n+1)}(x)$ instead of $f'(x)$.

By combining these three cases, we have the general theorem.

Tsuji [1] has generalized the lemma, Theorem 2 and Theorem 6, in that "pole" is replaced by "algebraic singularity."

[1] Japanese Journal of Mathematics, vol. iii, no. 2 (1926), p. 69. Generalizations of Theorems 2 and 6 have also been given by Polya, C. R., t. 184 (1921), p. 502, who proved that a series having the property (5) cannot have singularities of the same kind as those of a linear differential equation of Fuchsian type.

For other proofs and different generalizations of the lemma, Theorem 2 and Theorem 6, see Ostrowski, Jahresbuch der deutschen Math.-Vereinigung, Bd. 35 (1926), 9–12 Heft, p. 269, and Obrechkoff, Comptes Rendus, t. 184 (1927), p. 271.

24. THEOREM 7: *If, for the series* (1),

$$\overline{\lim_{n\to\infty}} \frac{\lambda_{n+1}}{\lambda_n} = \infty, \tag{18}$$

the function defined by the series has as singularities only unbounded continua.

We state, without proof, two theorems on which the proof is based. The first is due to Ostrowski, the second to Weierstrass.

THEOREM 8:[1] *If the series* (1) *has the property* (18), i.e., *if there exists a partial sequence* $\{\lambda_{n_i}\}$ *such that*

$$\lim_{i\to\infty} \frac{\lambda_{n_i+1}}{\lambda_{n_i}} = \infty,$$

and if we let

$$S_i(x) = \sum_{n=1}^{n_i} a_{\lambda_n} x^{\lambda_n},$$

then the sequence $\{S_i(x)\}$ *converges uniformly in each closed region interior to the region of existence of* $f(x)$; *moreover,* $f(x)$ *is uniform.*

THEOREM 9:[2] *If a sequence of functions* $f_n(x)$ *holomorphic in a closed region* D *is uniformly convergent on the boundary, then the sequence converges uniformly in the closed region* D. *The limiting function* $F(x)$ *is holomorphic within* D, *and* $F^{(k)}(x) = \lim_{n\to\infty} f_n^{(k)}(x)$.

LEMMA: *Let* E *be a bounded closed set, not a Cantor continuum. Then there exist two regions without common points, and at a distance not zero from each other, such that within each region is at least one point of* E, *and such that each point of* E *belongs to at least one of the regions.*

Construct about each point of E a circle of radius ϵ.

[1] Abh. aus dem Math. Seminär der Hamburgischen Universität (1922), p. 327.
[2] Montel, *Series des polynomes*, Paris (1910), p. 16.

There is a finite number of these circles, $C_1, C_2, \cdots, C_k$, such that each point of E is interior to at least one of them. Denote by D_ϵ the region composed of the interior points of these circles.

Since E is not a continuum, ϵ may be taken so small that D_ϵ is not connected. Otherwise each pair of points of E can be joined by a polygonal line of a finite number of segments, each of length less than 4ϵ, and such that each vertex is in E. Hence E is a Cantor continuum, contrary to hypothesis. Hence $D\epsilon$ consists of at least two connected regions.

There is at least one of these regions which occludes none of the others from the point at ∞. Let this be the first region. The other regions may be connected by polygonal lines which have no point in common with the first region, and are therefore at a finite distance from it. These can be covered by a finite number of overlapping circles of radius less than $\frac{\delta}{2}$, and with centers on these lines; these circles, together with those not of the first region, constitute the second region.

COROLLARY: *If, for every ϵ, a circle of radius ϵ is described about each point of a closed set E, and if, for every ϵ, the region formed by application of the Borel-Lebesgue theorem is connected, it is a continuum.*

THEOREM 10: *Given a closed curve C. Let $C_1, C_2, \cdots$ be a set of polygonal lines within C, all having a common vertex P. Suppose that each C_i has a vertex P_i such that $\lim_{i\to\infty} \eta_i = 0$, where η_i is the distance from P_i to C. Denote by ϵ_i the length of the longest segment of C_i, and suppose $\lim_{i\to\infty} \epsilon_i = 0$. Let E be the set of vertices of the C_i. Then E' is a continuum which contains P and a point of C.*

We note first that E' is not a null set, and is closed; further that it contains the point P, and some point of C. Describe a circle of radius η about each point of E'. Then there exists a finite number of these circles such that each point of E' is interior to at least one of them. Let $T\eta$ be the region composed of all of these circles. This region must contain all the vertices of all the C_i, $i > i_0$, where i_0 is sufficiently large. In fact, if there were a vertex P_i, outside or on the boundary of $T\eta$, for some i arbitrarily large, the points P_i would have a limiting point P', outside or on the boundary of $T\eta$, which is contrary to hypothesis.

The set E' is chained. For if A and B are two of its points, there will be a C_A of sides $< 2\,\eta$, of which one vertex is distant from A by less than $2\,\eta$, and a C_B of sides $< 2\,\eta$, of which one vertex is distant from B by less than $2\,\eta$; moreover, every vertex of C_A will be distant by less than $2\,\eta$ from some point of E', and similarly for C_B, if these are C_i, $i > i_0$, and C_A and C_B both contain P; this proves the statement.

Since E' is closed and chained, it is a continuum.

We proceed to the proof of Theorem 7. Let E denote the set of singularities of $f(x)$. Since, by Theorem 8, $f(x)$ is uniform, E is closed.

Moreover, E is perfect. For suppose an arbitrary point P_0 of E is not a limiting point. With P_0 as center describe a circle C_0 within which P_0 is the only singularity. Let C_1 be a smaller circle concentric with C_0. Within the ring thus formed, $f(x)$ is regular. By Theorem 8, the sequence

$$S_i(x) = \sum_{n=1}^{n_i} a_{\lambda_n} x^{\lambda_n}$$

converges uniformly in the closed ring, hence, by Theorem 9, within C_1, and consequently at P_0, which contradicts the hypothesis.

With center at an arbitrary point P_0 of E, describe a circle C_R of radius R, arbitrarily large. Denote by $\mathcal{E}$ the closed set of points composed of the circumference of C_R plus those points of E which lie within C_R.

The set $\mathcal{E}$ is a Cantor continuum. Otherwise there will be a closed curve K, lying within C_R, such that there is at least one point of $\mathcal{E}$ exterior to K, at least one point of $\mathcal{E}$ interior to K, and no point of $\mathcal{E}$ on K. Hence we can construct a ring surrounding K, lying within C_R, and having the same property. Then $S_i(x)$ converges uniformly within K, so that the points of $\mathcal{E}$ lying within K are regular.

We wish to prove that any point P of E belongs to a Cantor continuum P_C which is not bounded. Choose R large enough so that P is an interior point of the circle C_R. Let Q be a point on the circumference. Then P and Q are points of $\mathcal{E}$ and can be joined by a polygonal line whose vertices belong to $\mathcal{E}$:

$$P = P_1^i,\ P_2^i, \cdots, P_n^i, \cdots, P_k^i = Q,$$

where

$$\overline{P_j^i P_{j+1}^i} < \epsilon_i,\ j = 1, 2, \cdots, k-1,$$

and where $P_1^i, P_2^i, \cdots, P_n^i$ are all within and not on C_R, and P_n^i is the last such vertex. Then a point P of E may be joined to some point P' of E on C_R by a continuum consisting of points of E. The theorem is therefore proved, since P and R are arbitrary.

25. We have seen that a series for which the condition

$$\lim_{i\to\infty} (\lambda_{n_i+1} - \lambda_{n_i}) = \infty$$

is verified, has, on the circle of convergence, at least one singularity which is not a pole. But *a priori* there may also be poles on the circle of convergence. In this connection the following theorem is useful:

THEOREM 11: *If the series* $\Sigma a_n x^{\lambda_n}$ *represents a function having exactly one singularity on the circle of convergence, then the series* $\Sigma b_n x^{\lambda'_n}$, *where the* b_n *are arbitrary, can not have a pole with principal part*

$$\frac{A_p}{(x - x_0)^p}.$$

In particular, there are no simple poles on the circle of convergence.[1]

The proof is based on the following theorem:

THEOREM 12: *If, corresponding to the series*

$$\sum_{n=0}^{\infty} x^{\lambda_n},$$

we have a series

$$\sum_{n=0}^{\infty} c_n x^{\lambda'_n},$$

with unit radius of convergence, having the property that the series

$$\sum_{m=0}^{\infty} d_m x^m = \sum_{n=0}^{\infty} c_n x^{\lambda'_n} + \sum_{n=0}^{\infty} x^{\lambda_n},$$

where $d_{\lambda'_n} = c_n$, $d_{\lambda_n} = 1$, *is regular at the point* 1, *then the function*

$$\psi(x) = \sum_{n=0}^{\infty} a_n x^{\lambda_n},$$

where the a_n *are subject only to the condition that the series shall have unit radius of convergence, has at least two singular points on the circle of convergence.*

[1] Polya has generalized this theorem by showing that on the circle of convergence the function can have neither an algebraico-logarithmic point nor an isolated singular point about which the function remains uniform. Comptes Rendus, t. 184 (1927), p. 504. See also Gergen, American Journal of Mathematics, vol. 49 (1927), p. 407, for a generalization by means of "generalized lacunae," a concept introduced by the author. Bull. de la Soc. Math. de France, t. 53 (1925), p. 235.

This theorem, as we have stated, implies the preceding. Otherwise there will exist a series

$$\sum_{n=0}^{\infty} a_n x^{\lambda_n}$$

having one and only one singularity, and a corresponding series

$$\sum_{n=0}^{\infty} b_n x^{\lambda' n} \tag{19}$$

having a pole with principal part

$$\frac{A_p}{(x - x_0)^p}.$$

Integrate (19) $p - 1$ times. The resulting series,

$$\sum_{n=0}^{\infty} b'_n x^{\lambda' n + p - 1},$$

which we assume to have unit radius of convergence, will have a simple pole at a point $x_0 = e^{i\phi}$. The series

$$\frac{1}{x^{p-1}} \sum_{n=0}^{\infty} b'_n e^{ni\phi} x^{\lambda' n + p - 1} = \sum_{n=0}^{\infty} b''_n x^{\lambda' n} \tag{20}$$

has a simple pole at $x = 1$, and may therefore be written in the form

$$\frac{A}{1 - x} + \sum_{n=0}^{\infty} k_n x^n, \tag{21}$$

the last series being regular at $x = 1$. On the other hand, its radius of convergence is unity. For the series (20) has, on the unit circle, at least one singularity other than the pole $x = 1$. Otherwise we have an immediate contradiction. For $\overline{\lim}\ (\lambda'_{n+1} - \lambda'_n) > 1$, which means that there must be at least two poles on the circle of convergence. Hence every singularity of (20), other than the pole $x = 1$, is also a

singularity of the series in (21), that is, the latter series has a unit radius of convergence. We have the following relations:

$$-\frac{1}{A}\sum_{n=0}^{\infty} b_n'' x^{\lambda' n} = \sum_{n=0}^{\infty} b_n''' x^{\lambda' n} = \frac{-1}{1-x} - \sum_{n=0}^{\infty} \frac{k_n}{A} x^n,$$

$$\sum_{n=0}^{\infty} b_n''' x^{\lambda' n} + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{\lambda n} + \sum_{n=0}^{\infty} (1 + b_n''') x^{\lambda' n}$$

$$= -\frac{1}{A}\sum_{n=0}^{\infty} k_n x^n.$$

The series $\Sigma x^{\lambda n}$ accordingly satisfies the hypothesis of Theorem 5, where $c_n = 1 + b_n'''$. Hence the series $\Sigma a_n x^{\lambda n}$ has, contrary to hypothesis, at least two singularities on the circle of convergence.

We proceed to the proof of Theorem 12. If the theorem is false, there will exist a series $\Sigma a'_{\lambda_n} x^{\lambda n}$ with unit radius of convergence, having only one singular point, say $e^{i\phi}$, on the unit circle.

Denote by H the operation by which $\Sigma a_n b_n x^n$ is obtained from $\Sigma a_n x^n$ and $\Sigma b_n x^n$:

$$H\left(\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} a_n b_n x^n.$$

Since, by hypothesis, $\Sigma d_n x^n$ is regular at $x = 1$, we may write

$$H\left(\sum_{n=0}^{\infty} a'_{\lambda_n} x^{\lambda n} \sum_{n=0}^{\infty} d_n x^n\right) = \sum k_n x^n,$$

a series which, by Hadamard's theorem,[1] has, on the unit circle, singularities only of the form $\gamma = e^{i\phi} e^{i\theta}$, where $e^{i\theta}$

[1] The proof of Hadamard's theorem given in Chapter III assumes that the functions $f(x)$, $\phi(x)$ involved are uniform. But here all three series in question have radius of convergence 1. Hence it may be shown that the curves C, C_β may be drawn so that if x_0 is a regular point on the circumference for $f(x)$, and x_1 for $\phi(x)$, then the point $x_0 x_1$ will be included in the region of regularity for $F(z)$, from which the desired conclusion follows. [Editor.]

is a singularity of $\Sigma d_n x^n$. But we have

$$\sum_{n=0}^{\infty} k_n x^n = \sum_{n=0}^{\infty} a'_{\lambda_n} x^{\lambda_n}, \tag{22}$$

from the way in which the d_n are defined. Consequently

$$e^{i\phi} = e^{i\vartheta} e^{i\phi}, \quad e^{i\vartheta} = 1,$$

which contradicts the hypothesis that $x = 1$ is a regular point for $\Sigma d_n x^n$.

From the fact that the series (22) has at least two singularities on the unit circle, it follows that if $e^{i\phi_0}$ is one of them, there will be another, say $e^{i\phi_1}$, such that

$$e^{i\phi_0} = e^{i\vartheta} e^{i\phi_1}. \tag{23}$$

Thus for example, if the sequence $\{\lambda_n\}$ contains only a finite number of multiples of an integer p, the series $\Sigma a_n x^{\lambda_n}$ has at least two singular points on the circle of convergence.[1] For consider

$$-\frac{p}{1-x^p} = -p\sum_{n=0}^{\infty} x^{np} = -\frac{1}{1-x} + \phi(x),$$

where $\phi(x)$ is regular in the point $x = 1$. We may write

$$\begin{aligned}\sum_{n=0}^{\infty} d_n x^n &= -p\sum_{n=0}^{\infty} x^{np} + \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^{\lambda_n} + \sum_{n=0}^{\infty} (1-p)x^{\lambda'_n} \\ &= \phi(x),\end{aligned}$$

where np has been replaced by λ'_n. This function has poles at the points $e^{\frac{2\pi q i}{p}}$, $0 < q < p$, that is, the series Σx^{λ_n} satisfies the hypothesis of Theorem 12.

[1] See also Ostrowski: Jahresbuch der deutschen Math.-Verein., Bd. 35 (1926), 9–12 Heft, p. 269, and Jour. London Math. Soc., vol. 1, part 4, Oct., 1926, p. 236.

From the hypothesis concerning the λ_n, the θ of (23) must be of the form $\frac{2\pi qi}{p}$, $0 < q < p$. Hence if $e^{i\phi_0}$ is a singularity of $\Sigma a_{\lambda_n} x^{\lambda_n}$, there will be a singularity $e^{i\phi_1}$, of the same series such that

$$e^{i\phi_0} = e^{\frac{2\pi qi}{p}} e^{i\phi_1} \tag{24}$$

26. THEOREM 13: *Given a sequence* $\{\lambda_n\}$ *containing only a finite number of multiples of each* p_i *of an infinite sequence of prime numbers* $\{p_n\}$. *Then the series* $\Sigma a_{\lambda_n} x^{\lambda_n}$ *has an irreducible set of singularities on the circle of convergence.*

Without loss of generality, we may assume that $x = 1$ is a singular point. From (24) the set E of singularities on the unit circle consists of the points

$$e^{\frac{2\pi q_1 i}{p_1}}, \; e^{\frac{2\pi q_2 i}{p_2}}, \cdots, \; 0 < q_j < p_j,$$

and these points are distinct, since the p_j are prime numbers. Consequently E' is not a null set.

We shall prove that if $E^{(n)}$ is not a null set, and if to each $p_0^{(n)}$ of $E^{(n)}$, and for each p_j, there exists a number q_j such that

$$p_0^{(n)} e^{\frac{2\pi q_j^{(n)} i}{p_j}} = p_1^{(n)}, \tag{25}$$

then $E^{(n+1)}$ is not a null set, and for each $p_0^{(n+1)}$ we have

$$p_0^{(n+1)} e^{\frac{2\pi q_j^{(n+1)} i}{p_j}} = p_1^{(n+1)}.$$

The existence of $E^{(n+1)}$ follows at once from (25), since there is an infinity of distinct points $p_1^{(n)}$.

There exists a sequence of points $p_k^{(n)}$, $k = 1, 2, \cdots$ for which $\lim_{k \to \infty} p_k^{(n)} = p_0^{(n+1)}$. For an arbitrary j, the set of points

$$p_k^{(n)} e^{\frac{2\pi q_j^{(n)} i}{p_j}}, \; k = 0, 1, 2, \cdots$$

has at least one limiting point:

$$p_1^{(n+1)} = p_0^{(n+1)} e^{\frac{2\pi q_j^{(n+1)} i}{p_j}}.$$

In fact, since j is fixed, one of the factors $e^{\frac{2\pi q_j^{(n)} i}{p_j}}$ must be repeated an infinite number of times. This $q_j^{(n)}$ may be taken as $q_j^{(n+1)}$; hence the point $p_1^{(n+1)}$ thus given is distinct from $p_0^{(n+1)}$.

Generalizations of theorems similar to the preceding have been given by Ostrowski.[1] One of his theorems is the following:

If $f(x) = \Sigma a_n x^{\lambda_n}$, *where the sequence* $\{\lambda_n\}$ *does not contain any member of the form*

$$\left.\begin{array}{ll} l_1 + nq, & l_1 < q, \\ l_2 + nq, & l_2 < q, \\ \cdots\cdots & \cdots\cdots \\ l_k + nq & l_k < q, \end{array}\right\} \quad n = 1, 2, \cdots$$

where q *is a prime number, and where the integers* l_i *are distinct, then* $f(x)$ *has at least* $k + 1$ *singular points on its circle of convergence. If* $x = \alpha$ *is a singular point on the circle of convergence, then* $f(x)$ *has, on the circle of convergence, at least* k *other singularities* $\alpha_1, \alpha_2, \cdots, \alpha_k$ *of the form*

$$\alpha_j = \alpha e^{\frac{2\pi r_j i}{q}},$$

where r_j *is an integer,* $0 < r_j < q$.

27. We have seen the importance of determining whether the expression $\sqrt[m]{|D_{m,p+1}|}$ approaches 1 regularly or not. For a series $\Sigma a_n x^n$ having an infinity of lacunae, this regularity does not hold for $p = 1$, but since there is a partial sequence of coefficients $a_{n_1}, a_{n_2}, \cdots$ such that $\lim_{i\to\infty} \sqrt[n_i]{|a_{n_i}|} = 1$, we say that the sequence $\sqrt[n]{|a_n|}$ approaches 1 irregularly.

[1] *loc. cit.*

It is also natural to say that the series $\Sigma a_n x^n$ has more lacunae than the series $\Sigma b_n x^n$, or that $\sqrt[n]{|b_n|}$ approaches 1 more regularly than $\sqrt[n]{|a_n|}$ if, whatever the sequence of positive integers n_i, such that $\lim_{i\to\infty} \sqrt[n_i]{|a_{n_i}|} = 1$, it is also true that $\lim_{i\to\infty} \sqrt[n_i]{|b_{n_i}|} = 1$, but not *vice versa* (*i.e.*, there exists a sequence $\{n'_i\}$ such that $\lim_{i\to\infty} \sqrt[n'_i]{|b_{n'_i}|} = 1$, and $\lim_{i\to\infty} \sqrt[n'_i]{|a_{n'_i}|} < 1$).

This terminology has the advantage of permitting the formulation in a general manner of the law expressing the influence of the lacunae on the corresponding singularities, their nature and their number.

If the sequence $\sqrt[n]{|b_n|}$ approaches 1 irregularly so that the singularities of $\Sigma b_n x^n$ fail to satisfy an arbitrary property (A), then, from the foregoing theorems, there corresponds to the sequence $\sqrt[n]{|a_n|}$ the series $\Sigma a_n x^n$ whose singularities can not satisfy the property (B) less restrictive than (A), that is, every set of singularities satisfying (A) also satisfies (B), whereas there are sets satisfying (B) which do not satisfy (A). Briefly, the set of singularities satisfying (A) is in general contained in the set satisfying (B).

Conclusions of this kind are, moreover, the only ones that can be given when the hypotheses refer only to the absolute values of the coefficients. In particular, a theorem of Fatou states that it suffices to change the signs of an infinity of coefficients in order that the circle of convergence shall become a cut.

VI

THE DETERMINATION OF AN ANALYTIC FUNCTION BY A SEQUENCE OF COEFFICIENTS

28. The object of this chapter is to show the extent to which a function is determined by a partial sequence of its Taylor coefficients.

Consider a set of functions holomorphic within the unit circle, each function of the set having a given property (A). This property may give either the number or the nature of the singularities, or both. Then there will exist a sequence $\{n_i\}$ of integers corresponding to the set, *i.e.*, to property (A), such that if a sequence $\{a_{n_i}\}$ is given, the function $f(x)$ which has the a_{n_i} as its Taylor coefficients is defined to within a function whose radius of convergence exceeds 1, if (A) refers only to the singularities of $f(x)$ on the circle of convergence; if, on the other hand, (A) refers to the singularities of $f(x)$ in the entire plane, then $f(x)$ is determined to within an integral function. We consider now series with unit radius of convergence.

Two sequences $\{a_{m_j}\}$ and $\{a'_{m_j}\}$ for which

$$\overline{\lim_{m\to\infty}} \sqrt[m_j]{|a_{m_j} - a'_{m_j}|} < 1$$

are said to have the same *principal part.*

Suppose that we have a set of functions, and a sequence $\{n_i\}$ such that if the coefficients a'_{n_i}, a''_{n_i} of two functions of the set are the same, the sequences formed by the re-

maining coefficients a'_{n_j}, a''_{n_j}, will have the same principal part:

$$\overline{\lim_{n\to\infty}} \sqrt[n_j]{|a'_{n_j} - a''_{n_j}|} < 1. \tag{1}$$

We say then that the set of functions determines $\{a_{n_j}\}$ as a *functional* of $\{a_{n_i}\}$. If (A) refers to singularities in the entire plane, then (1) is replaced by

$$\lim_{m\to\infty} \sqrt[m_j]{|a'_{m_j} - a''_{m_j}|} = 0.$$

We shall see that when a property (A) referring to singularities on the circle of convergence is given, the set of functions whose singularities are characterized by (A) has the property that there exists a sequence $\{n'_i\}$ such that $\{a_{n_j}\}$ is a functional of $\{a_{n'_i}\}$. We shall verify this remark for several special properties (A). It will appear, as a consequence, that the following problem admits, at least theoretically, a solution:

Give the characteristics of the coefficients of a Taylor's series representing a function whose singularities have property (A).

Although the proofs of the theorems which follow are based on the theorems of the preceding chapter, the series considered are not necessarily lacunary.

29. Theorem 1: *Given the functions*

$$f_1(x) = \sum a_n x^n, \quad f_2(x) = \sum b_n x^n,$$

having only algebraic singularities on the circle of convergence. Let $\{n_r\}$, $\{k_r\}$ *denote two sequences of positive integers for which*

$$\lim_{r\to\infty} n_r = \infty, \quad \lim_{r\to\infty} k_r = \infty.$$

If $a_{n_r+p} = b_{n_r+p}$, $p = 0, 1, 2, \cdots, k_r$, *then the series for* $f_1(x) - f_2(x)$ *converges within a circle of radius greater than* 1.[1]

The interest of this theorem lies in the fact that $n_{r+1} - n_r$ may be arbitrarily great. In case $n_{r+1} - n_r = k_r + 1$ for all r, the theorem merely asserts the uniqueness of the function defined by a Taylor's series.

In order to prove the theorem, consider the sequence $r_1, r_2, \cdots$ for which $n_{r_l} + k_{r_l} + 1 < n_{r_l+1}$. This sequence may be assumed to be infinite, for otherwise $f_1 - f_2$ reduces to a polynomial.

Suppose the theorem false, so that

$$f_1(x) - f_2(x) = \sum c_n x^n$$

has its radius of convergence unity. If we denote by $\{m_j\}$ the sequence complementary to $\{n_i'\}$, we may write

$$\sum c_n x^n = \sum c_{m_j} x^{m_j},$$

since by hypothesis $c_{n_i'} = 0$ where $n_i' = n_r + p$. The m_j satisfy the hypothesis of Theorem 2, Chapter V:

$$\overline{\lim_{m \to \infty}} (m_{j+1} - m_j) = \infty.$$

Hence, by Tsuji's generalization of that theorem, the series $\Sigma c_{m_j} x^{m_j}$ has at least one non-algebraic singularity on its circle of convergence. This, however, is impossible. For, by hypothesis, f_1 and f_2 have only algebraic singularities on the unit circle, hence the same is true of $f_1 - f_2$.

[1] In Acta Math., t. 45 (1924), p. 129, the author has stated this theorem for the case in which f_1 and f_2 have only poles on the circle of convergence, basing the proof on Theorem 4 of the preceding chapter. As a result of Tsuji's generalization (*loc. cit.*) of the latter theorem, we may prove the more general theorem stated here.

If f_1 and f_2 have only algebraic singularities in the entire plane, then $f_1 - f_2$ is an integral function, provided

$$a_{n_r+p} = b_{n_r+p},\ p = 0, 1, \cdots, k_r,\ r = 1, 2, \cdots,$$

$$\lim_{r\to\infty} n_r = \infty,\ \lim_{r\to\infty} k_r = \infty.$$

For if $f_1 - f_2$ is not an integral function, it has a finite radius of convergence, and has only algebraic singularities on its circle of convergence.

We see, then, that if we consider the set of functions having only algebraic singularities on the circle of convergence, which is property (A) in this case, then the sequence $\{a_{m_j}\}$ is a functional of $\{a_{n'_r}\}$, where $n'_r = n_r + p, r = 1, 2, \cdots, p = 1, 2, \cdots, k_r$, and $\{m_j\}$ is complementary to $\{n'_i\}$.

30. Theorem 2: *If each of the sets of singularities of the functions*

$$f_1(x) = \sum a_n x^n,\ f_2(x) = \sum b_n x^n$$

is reducible; if $p_1, p_2, \cdots$ *is an increasing sequence of prime numbers, and if*

$$a_{np_r} = b_{np_r},\ r = 1, 2, \cdots,\ n = 1, 2, \cdots,$$

then the series $\Sigma c_n x^n$ *representing the function* $f_1 - f_2$ *converges within a circle of radius greater than* 1.

In the first place, we note that the set of singularities of $\Sigma c_n x^n$ is reducible. If the theorem is not true, the radius of convergence of this series is 1. Let $\Sigma c_n x^n = \Sigma c_{m_j} x^{m_j}$. Then the m_j satisfy the hypothesis of Theorem 13 of the preceding chapter. Hence we have the impossible conclusion that $\Sigma c_n x^n$ has an irreducible set of singularities on the unit circle.

For the set of functions having only reducible sets of singularities on the circle of convergence, the sequence $\{a_{m_j}\}$ is a functional of $\{a_{n'_i}\}$, where $\{n'_i\}$ is the sequence $\{np_r\}$ and $\{m_j\}$ is the complementary sequence.

As in Theorem 1, if the sets of singularities of f_1 and f_2 in the entire plane are reducible, then $f_1 - f_2$ is an integral function. Here property (A) is that of the reducibility of the sets of singularities.

31. We state without proof the following theorem:

THEOREM 3: *If $f_1(x)$ and $f_2(x)$ have sets of singularities in the entire plane such that no part of either set is a continuum, and if*

$$a_{n'_i} = b_{n'_i}, \quad n'_i = n_r + p, \quad r = 1, 2, \cdots, \quad p = 1, 2, \cdots, k_r,$$

$$\lim_{r\to\infty} \frac{k_r}{n_r} = \infty, \quad \lim_{r\to\infty} n_r = \infty,$$

then $f_1 - f_2$ is an integral function.

VII

ORDER OF SINGULARITIES[1]

32. The fractional derivative $D_x^\alpha f(x)$ is defined as follows:

$$D_x^0 f(x) = f(x),$$

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_k^x (x-z)^{-1-\alpha} f(z)\,dz, \quad \alpha < 0,$$

$$D_x^\alpha f(x) = \frac{d^p}{dx^p} D_x^{\alpha-p} f(x), \quad \alpha > 0,$$

where p is the smallest integer not exceeded by α.

The derivative thus defined for $\alpha < 0$ differs from the function

$$\frac{1}{\Gamma(-\alpha)} \int_0^x (x-z)^{-1-\alpha} f(z)\,dz$$

by a function which is regular except perhaps at 0 and k.

The Taylor expansion of $D_x^\alpha f(x)$ is obtained from the second of the above formulas. The series

$$(x-z)^{-1-\alpha} f(z) = (x-z)^{-1-\alpha} \sum a_n z^n$$

converges uniformly along the curve $C_{0,x}$;[2] integrating termwise, we have

[1] Hadamard, *loc. cit.* See pp. 154 ff.

[2] Provided $\alpha \leq -1$; if $-1 < \alpha < 0$, the termwise integration can be justified by a theorem of Osgood: *Lehrbuch der Funktionentheorie*, Leipzig (1912), vol. 1, p. 593; the S_{n_m} of that theorem is taken as

$$\int_0^{x-\frac{1}{n}} (x-z)^{-1-\alpha} \sum_{k=1}^{m} a_k z^k dz. \qquad \text{[Editor]}$$

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-z)^{-1-\alpha} \sum_{n=1}^{\infty} a_n z^n dz$$

$$= \frac{1}{\Gamma(-\alpha)} \sum_{n=1}^{\infty} a_n \int_0^x (x-z)^{-1-\alpha} z^n dz.$$

If we make the substitution $z = tx$, the integral

$$\int_0^x (x-z)^{-1-\alpha} z^n dz$$

becomes

$$x^{-\alpha} x^n \int_0^1 (1-t)^{-1-\alpha} t^n dt$$

$$= x^{-\alpha} x^n B(-\alpha, n+1)$$

$$= x^{-\alpha} x^n \frac{\Gamma(-\alpha)\,\Gamma(n+1)}{\Gamma(n+1-\alpha)},$$

from which we obtain

$$D_x^\alpha f(x) = x^{-\alpha} \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n x^n, \quad \alpha < 0.$$

If α is positive,

$$D_x^\alpha f(x) = \frac{d^p}{dx^p} \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha+p)} a_n x^{n-\alpha+p}$$

$$= \sum \frac{[\Gamma(n+1)](n+p-\alpha)(n+p-\alpha-1)\cdots(n-\alpha+1)}{\Gamma(n+1-\alpha+p)} a_n x^{n-\alpha}$$

$$= x^{-\alpha} \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n x^n.$$

33. The Hadamard operator H_x^α is defined as follows:

$$H_x^0 f(x) = f(x)$$

$$H_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_k^x (\log x - \log z)^{-1-\alpha} f(z)\, d \log z, \quad \alpha < 0,$$

$$H_x^\alpha f(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^p H_x^{\alpha-p} f(x), \quad \alpha > 0,$$

where p is defined as above.

If α is negative,

$$H_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \sum_{n=1}^{\infty} a_n \int_0^x \left(\log \frac{x}{z}\right)^{-1-\alpha} z^{n-1} dz.$$

In the integral

$$\int_0^x \left(\log \frac{x}{z}\right)^{-1-\alpha} z^{n-1} dz$$

place $z = tx$. The integral becomes

$$x^n \int_0^1 \left(\log \frac{1}{t}\right)^{-1-\alpha} t^{n-1} dt$$

$$= n^\alpha x^n \Gamma(-\alpha).^1$$

We have, then,

$$H_x^\alpha f(x) = \sum_{n=1}^{\infty} a_n n^\alpha x^n.$$

For $\alpha > 0$,

$$H_x^\alpha f(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^p H_x^{\alpha-p} f(x)$$

$$= \left(\frac{1}{x}\frac{d}{dx}\right)^p \sum_{n=1}^{\infty} a_n n^{\alpha-p} x^n$$

$$= \sum_{n=1}^{\infty} a_n n^\alpha x^n.$$

34. Let $f(x)$ be a function, real or complex, of the real variable x, $a \leq x \leq b$. If

$$\left| v \int_{a'}^{b'} f(x) \cos vx \, dx \right| < I, \left| v \int_{a'}^{b'} f(x) \sin vx \, dx \right| < I,$$

[1] For $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$. Let $v = e^{-\frac{x}{m}}$, $x = m \log \frac{1}{v}$. Then $\Gamma(\alpha) = m^\alpha \int_0^1 \left(\log \frac{1}{v}\right)^{\alpha-1} v^{m-1} dv$. [Editor]

for all v, where $a \leq a' < b' \leq b$, and where I is independent of a', b' and v, then $f(x)$ is said to be *of finite deviation* [1] in (a, b).

THEOREM 1: *A function $f(x)$ of limited variation in (a, b) is of finite deviation in (a, b).*

Let $Q = v\int_{a'}^{b'} f(x) \cos vx\, dx = \int_{a'}^{b'} f(x) d \sin vx.$

For a given v, divide (a', b') into subintervals by the points $x = \frac{k\pi}{v}$, using such integral values of k as give points in (a', b').

Consider a subinterval $\left(\frac{k}{v}\pi, \frac{k+1}{v}\pi\right)$, interior to (a', b'). Denote by M_k, m_k, respectively, the upper and the lower bound of $f(x)$ in this interval, and let $p_k = M_k - m_k$. Then, in this interval,

$$f(x) = m_k + \theta(x) p_k, \quad 0 \leq \theta \leq 1.$$

$$\int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} f(x) d \sin vx = \int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} m_k d \sin vx + \int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} \theta p_k d \sin vx.$$

The first integral on the right vanishes. Moreover,

$$\left| \int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} \theta\, p_k d \sin\, vx \right| < 2\, p_k.$$

Consider the left-hand interval, $\left(a', \frac{k_0}{v}\pi\right)$. If η and ϵ are positive constants, the latter being arbitrarily small, we have

$$\left| \int_{a'}^{\frac{k_0}{v}\pi} f(x)\, d \sin vx \right| \leq \int_{\frac{k_0-1}{v}\pi}^{\frac{k_0}{v}\pi} [|f(a')| + \eta] \left| d \sin vx \right|$$

$$= 2\,|f(a')| + 2\,\eta$$

$$< 2\,|f(a')| + \epsilon,$$

[1] Hadamard uses the term *écart fini*.

the last inequality being valid for $v > v_0$, since η can approach zero as n becomes infinite. Noting that a similar result is obtained for the right-hand interval we have, finally,

$$|Q| < 2\sum_{i=1}^{r} p_i + 2|f(a')| + 2|f(b')| + 2\epsilon,$$

where r is the number of subintervals interior to (a', b'). By hypothesis, Σp_i is bounded for all v, hence the same is true of Q.

A similar process, choosing properly the points of division of (a', b'), shows that the second integral appearing in the definition is also uniformly bounded.[1]

The converse is not true; in particular, an example has been given by H. E. Bray, namely, the function $x \sin \frac{1}{x}$. This function, which is not of limited variation, is of finite deviation. The proof of this statement depends on an investigation of the integrals:

$$\int_c^d \frac{\sin t}{\sqrt{t^2 - 4n}}\,dt, \quad \int_c^d \frac{\cos t}{\sqrt{t^2 - 4n}}\,dt, \quad 2\sqrt{n} \leq c < d,$$

$$\int_c^d \frac{\sin t}{\sqrt{t^2 + 4n}}\,dt, \quad \int_c^d \frac{\cos t}{\sqrt{t^2 + 4n}}\,dt.$$

In particular, every function which satisfies the Lipschitz condition is of finite deviation, since every function having a bounded derivative is of limited variation.

[1] A simpler proof of this theorem can be given on the basis of Stieltjes's theorem on integration by parts: *If $\phi(x)$ is continuous, and $\alpha(x)$ of limited variation, $a \leq x \leq b$, then*

$$\int_a^b \alpha(x)\,d\phi(x) = \alpha(x)\,\phi(x)\Big|_a^b - \int_a^b \phi(x)\,d\alpha(x). \qquad \text{[Editor]}$$

35. THEOREM 2: *Let* $f(x) = \Sigma a_n x^n$, *with* $R = 1$. *If* $\Sigma |a_n|$ *converges, and if* $\lim_{n\to\infty} n |a_n| \log n = 0$, *then* $f(x) = f(e^{i\phi}) = F(\phi)$ *is continuous and of finite deviation on the unit circle.*

The continuity of $F(\phi)$ is evident, since $\Sigma a_n e^{ni\phi}$ is dominated by the convergent series $\Sigma |a_n|$, and is accordingly uniformly convergent in every interval.

In order to show that I exists, independent of v, such that

$$\left| v \int_{a'}^{b'} F(\phi) \cos v\phi \, d\phi \right| < I, \quad \left| v \int_{a'}^{b'} F(\phi) \sin v\phi \, d\phi \right| < I,$$

it suffices to show that

$$J \equiv \left| v \int_{a'}^{b'} F(\phi) e^{vi\phi} d\phi \right| < I_0,$$

$$L \equiv \left| v \int_{a'}^{b'} F(\phi) e^{-vi\phi} d\phi \right| < I_0.$$

For a given v, we may write

$$J = \left| v \sum_{m=0}^{\infty} a_m \int_{a'}^{b'} e^{(m+v)i\phi} d\phi \right|$$

$$= \left| \sum_{m=0}^{\infty} a_m \frac{v}{m+v} [e^{(m+v)ib'} - e^{(m+v)ia'}] \right|$$

$$\leq \sum_{m=0}^{\infty} |a_m| \left| \frac{v}{m+v} \right| e^{(m+v)ib'} - e^{(m+v)ia'} \Big|$$

$$\leq 2 \sum_{m=0}^{\infty} |a_m|$$

$$= 2S,$$

and consequently J is uniformly bounded.

$$L = \left| v \sum_{m=0}^{\infty} a_m \int_{a'}^{b'} e^{(m-v)i\phi} d\phi \right| \tag{1}$$

$$= \left| \sum_{m=0}^{\infty} a_m \frac{v}{m-v} [e^{(m-v)ib'} - e^{(m-v)ia'}] \right|,$$

if v is not an integer; otherwise

$$L = \left| \sum_{m=0}^{\infty}{}' a_m \frac{v}{m-v} [e^{(m-v)ib'} - e^{(m-v)ia'}] + v a_v (b' - a') \right|,$$

where the prime indicates that the $(v+1)$-th term is omitted in the summation.

In the first case,

$$L \leq \sum_{m=0}^{\infty} |a_m| \frac{2v}{|m-v|},$$

and in the second case,

$$L \leq \sum_{m=0}^{\infty}{}' |a_m| \frac{2v}{|m-v|} + 2\pi |a_v|$$

Choose k and K so that $0 < k < 1 < K$. Consider, in (1), all the subscripts m for which $m \leq kv$. Each corresponding term is less than $\frac{2|a_m|}{1-k}$, so that

$$\sum_{m \leq kv} |a_m| \frac{2v}{|m-v|} < \frac{2S}{1-k}.$$

Similarly,

$$\sum_{m \geq Kv} |a_m| \frac{2v}{|m-v|} < \frac{2S}{K-1}.$$

Take next all the m's for which $kv < m < v$.

$$\sum_{kv < m < v} |a_m| \frac{2v}{|m-v|} \leq 2v |a_l| \left(1 + \frac{1}{2} + \cdots + \frac{1}{h}\right), \tag{2}$$

where a_l is the coefficient of greatest absolute value for $kv < m \leq Ev$, and h the number of such integers m.

Since l is one of the indices occurring in the sum, the inequalities

$$k < \frac{l}{v} \leq 1$$

are verified. Hence

$$2v\,|a_l|\left(1+\frac{1}{2}+\cdots+\frac{1}{h}\right) < \frac{2l}{k}\,|a_l|\left(1+\frac{1}{2}+\cdots+\frac{1}{h}\right).$$

But we have

$$\begin{aligned} h &= E[(1-k)v] - 1 \\ &= (1-k)v - 1 - \epsilon_v, \quad 0 \leq \epsilon_v < 1, \\ &= \left(1-k-\frac{1+\epsilon_v}{v}\right)v \\ &= \rho_v v, \end{aligned}$$

where $$\rho_v = 1 - k - \frac{1+\epsilon_v}{v}.$$

Moreover,

$$\lim_{v\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{v\rho_v}}{\log(v\rho_v)} = 1, \tag{3}$$

since $\log v\rho_v = \log v + \log \rho_v$, and $\lim_{v\to\infty} \rho_v = 1 - k \neq 0$, so that $h = v\rho_v$ becomes infinite with v.

On the other hand,

$$1 \leq \frac{v}{l} < \frac{1}{k} = \delta,$$

so that

$$0 \leq \log v - \log l < \log \delta.$$

Hence, for all v, we have

$$|\log v - \log l| < M, \tag{4}$$

and we may therefore replace $\log v\rho_v$ in (3) by $\log l$:

$$\lim_{v\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{h}}{\log l} = 1.$$

Now, given ϵ, there exists a number v_ϵ such that for $v > v_\epsilon$ we have, from (2),

$$\sum_{kv<m<v} |a_m| \frac{2v}{|m-v|} < \frac{2l}{k} |a_l| (1+\epsilon) \log l. \tag{5}$$

By hypothesis $l\,|\,a_l\,|\,\log l$ approaches 0 as l becomes infinite, hence as v becomes infinite, because of (4). And since the right hand side of (5) is bounded for $v < v_\epsilon$, having only a finite number of terms, it follows that this expression is bounded for all v.

There exists, by hypothesis, a number L_0 such that $L_0\,|\,a_{L_0}\,|\log L_0$ exceeds any other $l\,|\,a_l\,|\log l$. We may therefore write

$$\sum_{kv<m<v} |a_m| \frac{2v}{|m-v|} < \mu L_0 |a_{L_0}| \log L_0,$$

where μ is a certain constant.

In a similar way it may be shown that

$$\sum_{1+Ev<m<Kv} |a_m| \frac{2v}{|m-v|} < \mu' L_0 |a_{L_0}| \log L_0.$$

In case v is an integer there remains to be examined only the term $va_v(b'-a')$. This expression is bounded, for it approaches zero as v becomes infinite.

If v is not an integer, consider

$$a_{Ev} \frac{v}{Ev-v} [e^{(Ev-v)ib'} - e^{(Ev-v)ia'}] \tag{6}$$

and

$$a_{Ev+1} \frac{v}{Ev+1-v} [e^{(Ev+1-v)ib'} - e^{(Ev+1-v)ia'}]. \tag{7}$$

We have

$$\left| e^{(E\upsilon-\upsilon)ib'} - e^{(E\upsilon-\upsilon)ia'} \right| = \left| 2\, ie^{(E\upsilon-\upsilon)i\frac{a'+b'}{2}} \sin \frac{(E\upsilon-\upsilon)(b'-a')}{2} \right|$$

$$< (E\upsilon-\upsilon)(a'+b').$$

Hence the expression (6) is, in absolute value, less than

$$|a_{E\upsilon}|\upsilon(b'+a') \leq 2\,\pi\upsilon|a_{E\upsilon}|$$

if we take $b'+a' \leq 2\,\pi$. The last expression approaches zero as υ becomes infinite.

Similarly the quantity $2\,\pi\upsilon\,|a_{E\upsilon+1}|$ is an upper bound for the expression (7).

Combining the foregoing results, we obtain

$$L < \frac{2S}{1-k} + \frac{2S}{K-1} + \mu L_0|a_{L_0}|\log L_0 + \mu' L_0|a_{L_0}|\log L_0 + 2M, \quad (8)$$

where M exceeds the largest of the numbers of the form $\upsilon\,|\,a_\upsilon\,|\,(b'-a')$, $2\,\pi\upsilon\,|\,a_{E\upsilon}\,|$, $2\,\pi\upsilon\,|\,a_{E\upsilon+1}\,|$. Since each of these expressions approaches zero as υ becomes infinite, the number M exists.

COROLLARY: *The deviation of $f(x)$ satisfies the inequality*

$$L < k_1 S + k_2 L_0|a_{L_0}|\log L_0 + k_3 l_0|a_{l_0}|, \quad (8')$$

where k_1, k_2, k_3 are numbers not depending on the particular series, and $l_0\,|\,a_{l_0}\,|$ is the maximum value of $k\,|\,a_k\,|$ for all k, the numbers S and L_0 having their previous significance.

On the other hand, we have also the following theorem:

THEOREM 3: *Given $f(x) = \Sigma a_n x^n$, $R = 1$. If $f(e^{i\phi}) = F(\phi)$ is continuous and of finite deviation on the unit circle, then for every $\epsilon > 0$,*

(*a*) $\sum \frac{|a_n|}{n^\epsilon}$ *converges, and*

(*b*) $\lim_{n\to\infty} n^{1-\epsilon}|a_n|\log n = 0$.[1]

[1] Or, what is the same thing,

$$\lim_{n\to\infty} n^{1-\epsilon}|a_n| = 0 \text{ for every } \epsilon > 0.$$ [EDITOR]

Construct a circle C_r with center at the origin and radius $r < 1$. Then

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(x)dx}{x^{n+1}}.$$

On the other hand, since

$$\lim_{r \to 1} f(re^{i\phi}) = f(e^{i\phi})$$

uniformly for all ϕ, we have

$$\lim_{r \to 1} \int_{C_r} \frac{f(x)}{x^{n+1}} dx = \int_{C} \frac{f(x)}{x_{n+1}} dx,$$

where C is the unit circle. Consequently

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) e^{-ni\phi} d\phi.$$

Since $f(e^{i\phi})$ is of finite deviation, L_0 exists such that

$$\left| n \int_0^{2\pi} f(e^{i\phi})\, e^{-ni\phi} d\phi \right| < L_0,$$

and therefore

$$n \mid a_n \mid < L, \qquad L = \frac{L_0}{2\pi},$$

$$\frac{\mid a_n \mid}{n^\epsilon} < \frac{L}{n^{1+\epsilon}},$$

so that $$\sum \frac{\mid a_n \mid}{n_\epsilon}$$ converges.

In order to prove the second part of the theorem, we need only note that

$$n \frac{\mid a_n \mid}{n^\epsilon} \log n < \frac{L \log n}{n^\epsilon},$$

and that the limit of the right-hand side is zero.

36. The *order* of $f(x)$ on an arc of the unit circle is the number ω such that $H^{-(\omega+\epsilon)} f(e^{i\phi})$ is continuous and of finite deviation on the arc, whereas, for $H^{-(\omega-\epsilon)} f(e^{i\phi})$, at least one of these properties fails, that is, $H^{-(\omega-\epsilon)} f$ is either discontinuous or not of finite deviation, or both discontinuous and not of finite deviation.

The order of $f(e^{i\phi})$ on any arc, $a \leq \phi \leq b$, of the unit circle is clearly not greater than the order of $f(e^{i\phi})$ in an arc containing (a, b). The *order of* $f(e^{i\phi})$ *in the point* ϕ_0 is defined as the lower bound of the orders of $f(e^{i\phi})$ in all the intervals containing ϕ_0. If $f(e^{i\phi})$ is of order ω on the circumference, there is at least one point on the circumference at which the order is ω; there is no point on the circumference at which the order exceeds ω.

THEOREM 4: *The order of* $f(x)$ *on the circle of convergence is given by*

$$\omega = \overline{\lim_{n\to\infty}} \frac{\log |a_n|}{\log n} + 1.$$

We have, by definition, that for all except a finite number of values of n,

$$1 + \frac{\log |a_n|}{\log n^1} = \frac{\log (n |a_n|)}{\log n} < \omega + \epsilon,$$

with $\epsilon > 0$ arbitrary. Hence

$$n |a_n| < n^{\omega+\epsilon},$$

$$\frac{|a_n|}{n^{\omega+2\epsilon}} < \frac{1}{n^{1+\epsilon}}.$$

The series $\sum \frac{|a_n|}{n^{\omega+2\epsilon}}$ accordingly converges.

Moreover, from the preceding inequality, we have, for n sufficiently large,

$$\frac{n |a_n|}{n^{\omega+2\epsilon}} \log n < \frac{1}{n^{\frac{\epsilon}{2}}},$$

$$\lim_{n\to\infty} \frac{n |a_n|}{n^{\omega+2\epsilon}} \log n = 0.$$

Hence the series $\sum b_n x^n$, where $b_n = \frac{a_n}{n^{\omega+2\epsilon}}$, satisfies the hypothesis of Theorem 2. Consequently $H^{-(\omega+2\epsilon)} f(e^{i\phi})$ is continuous and of finite deviation on the unit circle.

On the other hand, there is an infinity of values of n such that

$$\frac{\log (n \mid a_n \mid)}{\log n} > \omega - \epsilon,$$

$$\frac{\mid a_n \mid}{n^{\omega-\epsilon}} > \frac{1}{n},$$

$$\frac{n \mid c_n \mid}{n^{\epsilon}} \log n > \log n,$$

where $c_n = \frac{a_n}{n^{\omega-2\epsilon}}$. The last inequality shows that $H^{-(\omega-2\epsilon)} f(x) = \Sigma c_n x^n$ is not both continuous and of finite deviation; otherwise we should have, by Theorem 3,

$$\overline{\lim_{n\to\infty}}\, n^{1-\epsilon} \mid c_n \mid \log n = 0.$$

We have seen that the expression

$$\overline{\lim_{n\to\infty}} \frac{\log \mid a_n \mid}{n} = -\log R$$

gives the absolute value of at least one singular point. It is natural to inquire whether similar formulas yield information regarding the kind of singularities on the circle of convergence. The answer is affirmative. In fact, the preceding theorem shows that the value of the expression

$$\overline{\lim_{n\to\infty}} \frac{\log \mid a_n \mid}{a_n}$$

determines the order of the function on its circle of convergence.

37. We have often employed the following fact: If x_0 is an isolated essential singularity of the function $f(x)$ about which the function remains uniform, then we may write

$$f(x) = f_1(x) + f_2(x),$$

where x_0 is the only singularity of $f_1(x)$, and $f_2(x)$ is holomorphic in x_0. We may state a similar fact for the separation of the arcs on which the function has a given order.

THEOREM 5: *Let $f(x)$ be a function having the order ω on the closed arc Γ of the unit circle (of convergence). Then, given $\epsilon > 0$, we may write*

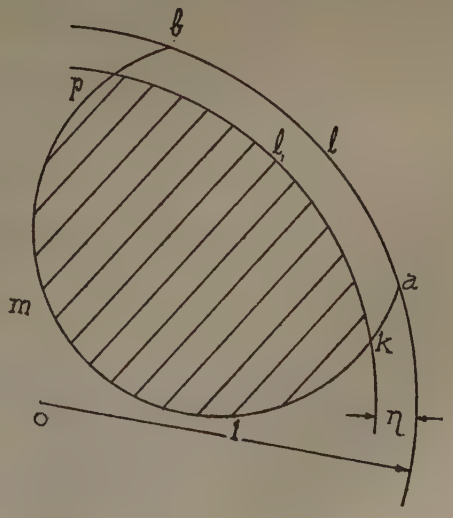
$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ is of order not greater than $\omega + \epsilon$ on the entire unit circle, and $f_2(x)$ is holomorphic on Γ, except perhaps at a and b.

By hypothesis, the function

$$\phi(x) = H_x^{-\omega-\epsilon} f(x)$$

is continuous and of finite deviation on Γ.



In the figure, Γ is the arc ab. Draw a circle passing through a and b, and having its center within the unit circle. With O as center, draw a circle of radius $1 - \eta$. Let T denote the shaded region.

For every point of T, we may write $\phi(x) = \phi_1(x) + \phi_2(x)$, where

$$\phi_1(x) = \frac{1}{2\pi i}\int_{klp} \frac{\phi(z)}{z-x}\,dz,$$

$$\phi_2(x) = \frac{1}{2\pi i}\int_{pmk} \frac{\phi(z)}{z-x}\,dz.$$

If we let η approach zero, $\phi_1(x)$ and $\phi_2(x)$ vary continuously, since $\phi(x)$ is continuous. We may therefore write

$$\phi_1(x) = \frac{1}{2\pi i}\int_{alb} \frac{\phi(z)}{z-x}\,dz,$$

$$\phi_2(x) = \frac{1}{2\pi i}\int_{bma} \frac{\phi(z)}{z-x}\,dz.$$

By writing

$$\frac{1}{z-x} = \frac{1}{z}\frac{1}{1-\frac{x}{z}} = \frac{1}{z}\left\{1 + \frac{x}{z} + \frac{x^2}{z^2} + \cdots\right\},$$

we have

$$\phi_1(z) = \sum k_n x^n,$$

where

$$k_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{\phi(x)dx}{x^{n+1}}.$$

Since $\phi(x)$ is of finite deviation on Γ,

$$\left|k_n\right| = \frac{1}{2\pi}\left|\int_{\Gamma} \phi(x)[\cos(n+1)\theta - i\sin(n+1)\theta]\right.$$
$$\left.[-\sin\theta + i\cos\theta]\,d\theta\right| < \frac{L}{n}, \qquad (9)$$

where $x = \cos\theta + i\sin\theta$, and L is a constant.

On the other hand, we have, by definition of ϕ, ϕ_1, and ϕ_2,

$$f(x) = H_x^{\omega+\epsilon}\phi(x)$$
$$= H_x^{\omega+\epsilon}\phi_1(x) + H_x^{\omega+\epsilon}\phi_2(x). \qquad (10)$$

We may write

$$H_x^{\omega+\epsilon}\phi_1(x) = \sum l_n x^n,$$

where $l_n = k_n n^{\omega+\epsilon}$. Hence, from (9),

$$|l_n| < Ln^{-1+\omega+\epsilon},$$

and the function $H_x^{\omega+\epsilon}\phi_1(x)$ is of order not greater than $\omega + \epsilon$ on Γ, by Theorem 4.

On the other hand, $\phi_2(x)$ is developable in a Taylor's series, since it is the difference of two Taylor's series. Moreover, $\phi_2(x)$ is holomorphic on Γ, except perhaps at a and b, since it is holomorphic in every point which is not on the arc amb. Hence $H_x^{\omega+\epsilon}\phi_2(x)$ is also holomorphic on Γ.

We see therefore from (10) that $f(x)$ is represented in the desired form.

38. We may now prove two theorems which are useful in investigating the behavior of a function in the neighborhood of a singular point.

THEOREM 6: *Suppose $f(x)$ has on the closed arc Γ of the circle of convergence the order ω_1 less than a positive number ω. Let ϕ_1, ϕ_2 be two quantities such that $0 \leq \phi_1 < \phi_2 < 2\pi$, and such that Γ contains the arc $\phi_1\phi_2$ in its interior. Then, as $r \to 1$,*

(*a*) *the quantity*

$$(1 - r)^\omega f(re^{i\phi})$$

approaches zero uniformly with respect to ϕ, provided $\phi_1 \leq \phi \leq \phi_2$; and

(*b*) *the quantity*

$$(1 - r)^\omega I_r,$$

where I_r is the deviation of $f(re^{i\phi})$ on the arc $\phi_1\phi_2$ of the circle of radius r, also approaches zero.

We may assume that $f(x)$ is of order ω on the whole circle, which involves no loss of generality. In fact, from the preceding theorem, we may write $f(x) = f_1(x) + f_2(x)$, where

$f_1(x)$ is of order $< \omega$ on the entire circle, and where $f_2(x)$ is holomorphic on Γ, except at the end points; the theorem is evidently true for $f_2(x)$.

We have, by Theorem 3,

$$\lim_{n\to\infty} \frac{a_n}{n^{\omega-1}} = 0.$$

Now, since

$$\lim_{n\to\infty} \frac{\Gamma(n+\omega)}{\Gamma(n+1)n^{\omega-1}} = 1,^{1}$$

and since

$$\frac{1}{(1-x)^{\omega}} = \sum d_n x^n,$$

where

$$d_n = \frac{\Gamma(n+\omega)}{\Gamma(\omega)\Gamma(n+1)},$$

it follows that

$$\lim_{n\to\infty} \frac{a_n \Gamma(\omega)\Gamma(n+1)}{\Gamma(n+\omega)} = \lim_{n\to\infty} \frac{a_n}{d^n} = 0.$$

Hence, for an arbitrary $\epsilon > 0$,

$$| a_n | < \frac{\epsilon}{2} d_n, \quad n > n_\epsilon.$$

If we add to $\frac{1}{(1-x)^{\omega}}$ a polynomial,

$$P(x) = \sum_{i=1}^{n_\epsilon} A_i x^i, \quad A_i > | a_i |,$$

we have, *a fortiori*,

$$| a_i | < \frac{\epsilon}{2} d_i + A_i, \quad i = 1, 2, \cdots, n_\epsilon,$$

$$| a_0 + a_1 r e^{i\phi} + \cdots + a_{n_\epsilon} r^{n_\epsilon} e^{n_\epsilon i\phi} | <$$

$$\frac{\epsilon}{2} (d_0 + d_1 r + \cdots + d_{n_\epsilon} r^{n_\epsilon}) + P | (x) |,$$

[1] As may be obtained from the approximation formula:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\, n^n e^{-n}. \qquad \text{[Editor]}$$

and

$$| a_{n_\epsilon+1} r^{n_\epsilon+1} e^{(n_\epsilon+1)i\phi} + \cdots \cdot | < \frac{\epsilon}{2} d_{n_\epsilon+1} r^{n_\epsilon+1} + \cdots,$$

so that

$$| f(x) | < \frac{\epsilon}{2} \sum_{n=0}^{\infty} d_n r^n + | P(x) |$$

$$= \frac{\epsilon}{2} \frac{1}{(1-r)^\omega} + | P(x) |,$$

$$(1-r)^\omega | f(x) | < \frac{\epsilon}{2} + (1-r)^\omega | P(x) |,$$

where $x = re^{i\phi}$. The first part of the theorem follows at once.

For the proof of the second part of the theorem, it suffices, in view of the Corollary of Theorem 2, to show that

$$\lim_{r\to 1} (1-r)^\omega (k_1 S_r + k_2 L_r | a_{L_r} | \log L_r r^{L_r} + k_3 l_r | a_{l_r} | r^{l_r}) = 0.$$

The reasoning of the first part of the theorem shows that

$$\lim_{r\to 1} (1-r)^\omega S_r = 0.$$

If L_r as a function of r remains bounded, say $< m_0$, for all r such that $1 - \eta < r < 1$, where η is fixed, we have

$$\lim_{r\to 1} (1-r)^\omega L_r | a_{L_r} | \log L_r r^{L_r} = 0. \tag{11}$$

Choose ω' so that $\omega_1 < \omega' < \omega$. From Theorem 3, we have, for $m > m_0$,

$$| a_m | < \frac{m^{\omega'-1}}{\log m}, \tag{12}$$

and, placing $r = 1 - \epsilon$,

$$(1-r)^\omega m \log m | a_m | r^m < m^{\omega'} (1-\epsilon)^m \epsilon^\omega, \tag{13}$$

for $m > m_0$.

Consider (11) for the case in which L_r increases indefinitely as $r \to 1$, and in particular when $L_r > m_0$ as may be the case when r is sufficiently near 1.

We note that the expression $m^{\omega'}(1-\epsilon)^m\epsilon^\omega$ does not increase in passing from $m = n$ to $m = n+1$, provided

$$n > \frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1}.$$

For in order that $n^{\omega'}(1-\epsilon)^n\epsilon^\omega < (n+1)^{\omega'}(1-\epsilon)^{n+1}\epsilon^\omega$, it is necessary and sufficient that

$$\left(\frac{n+1}{n}\right)^{\omega'} > \frac{1}{1-\epsilon},$$

$$n < \frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1}.$$

The expression $m^{\omega'}(1-\epsilon)^m\epsilon^\omega$ will therefore attain its greatest value for $m = m_k$, where m_k, or perhaps $m_k + 1$, is given by the expression

$$E\left[\frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1}\right].$$

But we have

$$\frac{1}{z^\alpha - 1} = \frac{1}{\alpha}\,\frac{z}{z-1} + \theta(z),$$

where $\theta(z)$ is bounded in the neighborhood of $z = 1$, *i.e.*,

$$\left|\frac{1}{z^\alpha - 1} - \frac{1}{\alpha}\,\frac{z}{z-1}\right| < M,$$

for $|z-1| < \eta$, where η is sufficiently small.

Let $z = \frac{1}{1-\epsilon}$, $\alpha = \frac{1}{\omega'}$; then

$$\left|\frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1} - \frac{\omega'}{\epsilon}\right| < M_1$$

when $|\epsilon| < \eta_1$. Hence

$$\left| m_k - \frac{\omega'}{\epsilon} \right| < M_2, \quad |\epsilon| < \eta_1,$$

$$\frac{\omega'}{\epsilon} - M_2 < m_k < \frac{\omega'}{\epsilon} + M_2,$$

$$(1-\epsilon)^{m_k} < (1-\epsilon)^{\frac{\omega'}{\epsilon} - M_2} = (1-\epsilon)^{\frac{\omega'}{\epsilon}} C,$$

where $C = (1-\epsilon)^{-M_2} = \text{const.}$ Also

$$m_k^{\omega'} < \left(\frac{\omega'}{\epsilon} + M_2 \right)^{\omega'}$$

$$< \left(\frac{\omega'}{\epsilon} \right)^{\omega'} C_1^{\omega'},$$

where $C_1 = \text{const.} > 1 + \dfrac{M_2 \epsilon}{\omega'}$. From these inequalities and (13), we have, for $L_r > m_0$,

$$(1-r)^{\omega} L_r \log L_r |a_{L_r}| r^{L_r} < L_r^{\omega'} (1-\epsilon)^{L_r} \epsilon^{\omega}$$

$$\leq m_k^{\omega'} (1-\epsilon)^{m_k} \epsilon^{\omega}.$$

It suffices therefore to show that

$$\lim_{\epsilon \to 0} (1-\epsilon)^{\frac{\omega'}{\epsilon}} \left(\frac{\omega'}{\epsilon} \right)^{\omega'} \epsilon^{\omega} = 0, \tag{14}$$

from which (11) will follow.

The proof of (14) is immediate. For

$$\lim_{\epsilon \to 0} (1-\epsilon)^{\frac{\omega'}{\epsilon}} = e^{\omega'},$$

and, since $\omega > \omega'$,

$$\lim_{\epsilon \to 0} \epsilon^{\omega - \omega'} = 0.$$

It remains to be proved that

$$\lim_{r \to 1} (1-r)^{\omega} l_r |a_{l_r}| r^{l_r} = 0.$$

This statement is verified if l_r as a function of r remains bounded.

If l_r becomes infinite as r approaches 1, then

$$l_r|a_{l_r}| < l_r|a_{l_r}|\log l_r \leq L_r|a_{L_r}|\log L_r.$$

This completes the proof, since it has been shown that

$$\lim_{r\to 1}(1-r)^{\omega}L_r|a_{L_r}|\log L_r = 0.$$

THEOREM 7: *If, for* $\omega > 0$,

$$\lim_{r\to 1}(1-r)^{\omega}f(re^{i\theta}) = 0,$$

uniformly, for $\alpha \leq \theta \leq \beta$, *and*

$$\lim_{r\to 1}(1-r)^{\omega}I_r = 0,$$

where I_r *is the deviation of* $f(re^{i\theta})$ *on the arc* $(\alpha\beta)$, *the function* $f(x)$ *is of order* $\omega' < \omega$ *on the arc* $(\alpha\beta)$.

In determining the order of $f(x)$ we may consider $x^{-\omega}D_x^{-\omega}f(x)$ instead of $H_x^{-\omega}f(x)$, since if one of these functions is continuous and of finite deviation, so is the other.[1]

The expression

$$x^{-\omega''}D_x^{-\omega''}f(x) = F(x) = \frac{1}{\Gamma(\omega'')}\int_0^1 (1-t)^{\omega''-1}f(tx)\,dt,$$

where $\omega'' > \omega$, is finite and continuous in the region

$$r \leq 1, \quad \alpha \leq \theta \leq \beta, \quad x = re^{i\theta}.$$

For in this region we have, by hypothesis,

$$\left|f(tx)\right| < \frac{A}{(1-|tx|)^{\omega}} \leq \frac{A}{|1-t|^{\omega}}.$$

[1] Hadamard, *loc. cit.*, p. 158.

But since $\omega'' - \omega > 0$,

$$\left|\int_0^1 (1-t)^{\omega''-1} f(tx)\,dt\right| < A\int_0^1 (1-t)^{\omega''-\omega-1}\,dt,$$

so that for the values of x in question,

$$|F(x)| < \frac{A}{\Gamma(\omega'')}\int_0^1 (1-t)^{\omega''-\omega-1}\,dt.$$

Thus $F(x)$ is bounded in (α, β). Moreover it is continuous, since the integral converges uniformly.

By definition, the deviation of $F(x)$ is the upper bound of

$$n\left|\int_\alpha^\beta e^{\pm ni\theta} F(e^{i\theta})\,d\theta\right|$$

$$= \frac{n}{\Gamma(\omega'')}\left|\int_\alpha^\beta e^{\pm ni\theta}\int_0^1 (1-t)^{\omega''-1} f(te^{i\theta})\,dt\right|.$$

This integral may be regarded as a double integral, since the integrand is dominated by $A(1-t)^{\omega''-\omega-1}$. We may therefore reverse the order of integration:

$$\frac{1}{\Gamma(\omega'')}\int_0^1 \left\{(1-t)^{\omega''-1}\int_\alpha^\beta ne^{\pm ni\theta} f(te^{i\theta})\,d\theta\right\} dt.$$

But

$$\left|\int_\alpha^\beta ne^{\pm ni\theta} f(te^{i\theta})\,d\theta\right| < I_t,$$

where I_t is the deviation of $f(te^{i\theta})$ in (α, β). By hypothesis

$$I_t < \frac{B}{(1-t)^\omega},$$

where B is fixed. Consequently

$$\frac{n}{\Gamma(\omega'')}\int_\alpha^\beta e^{\pm ni\theta} F(e^{i\theta})\,d\theta < \frac{B}{\Gamma(\omega'')(\omega''-\omega)},$$

and $F(x)$ is of finite deviation on the arc (α, β). Since $\omega'' > \omega$, the theorem is proved.

It appears from the two preceding theorems that the order of $f(x)$ on (α, β), if positive, may be defined as the number ω such that the quantities $(1 - r)^{\omega+\epsilon} f(re^{i\theta})$ and $(1 - r)^{\omega+\epsilon} I_r$ remain bounded for an arbitrary ϵ as r approaches 1, whereas one or both of these expressions fails to remain bounded if ϵ is replaced by $-\epsilon$.

39. For use in a later chapter, we state without proof a theorem of Fabry.[1]

THEOREM 8: *If, for the series* $\Sigma a_{\lambda_n} x^{\lambda_n}$, *we have*

$$\lambda_{n+1} - \lambda_n > k\sqrt{\lambda_n \log \lambda_n},$$

where k is a positive constant, then every point on the circle of convergence has the same order, namely, the order of the function on the circle of convergence.

[1] Comptes Rendus, t. 151 (1910), p. 922.

VIII

FUNCTIONS HAVING ONLY ONE SINGULARITY

40. An important problem in the theory of Taylor's series is that of determining the conditions to be satisfied by the coefficients in order that the function defined by the series (with finite radius of convergence) shall have exactly one singularity in the entire plane. The problem was solved first by Leau.[1] Later Faber[2] established necessary and sufficient conditions. His theorem is the following:

THEOREM 1: *In order that the function*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad 0 < R < \infty,$$

have the point 1 *as its only singularity in the entire plane, it is necessary and sufficient that* $a_n = g(n)$, $n = 1, 2, \cdots$, *where* $g(z)$ *is an integral function such that*

$$|g(z)| < e^{r^s} \text{ for } r > r_0, \quad r = |z|.$$

Leau gave, as a sufficient condition, $a_n = g(n)$, where

$$|g(z)| < e^{r^{s+\epsilon}}, \text{ for } r > r_0, \quad 0 < s < 1.$$

The latter condition is more restrictive than the former. For we may write

$$e^{r^{s+\epsilon}} < (e^r)^{r^{s+2\epsilon-1}}$$

[1] Journal de Math., 5e série, t. v. (1899), p. 409.
[2] Math. Annalen, t. 57 (1903), p. 369.

and since $s < 1$, we have, for a suitably chosen α,

$$s + 2\epsilon - 1 < -\alpha < 0,$$

if ϵ is sufficiently small. Hence

$$|g(z)| < e^{r^{s-\alpha}} \text{ for } r > r_\epsilon.$$

If we take r'_ϵ greater than each of the numbers r_ϵ, $\left(\frac{1}{\epsilon}\right)^{\frac{1}{\alpha}}$, we have

$$|g(z)| < e^{\epsilon r} \text{ for } r > r'_\epsilon.$$

If, then, the condition of Leau is satisfied, so is that of Faber.

Assume first that it has been shown that there is no singularity except at the point $x = 1$, if the a_n satisfy the conditions of the hypothesis. If, also, $x = 1$ is a regular point, we must have $a_n = g(n) = 0$, $n = 1, 2, \cdots$, since, by Liouville's theorem, $f(x)$ reduces to a constant. Suppose $g(z)$ has a zero of order k at $z = 1$. Then

$$g_1(z) = \frac{g(z)}{(1-z)^k}$$

is also an integral function which satisfies the hypothesis of the theorem, and we have $g_1(n) = 0$, $n = 2, 3, \cdots$. The series $\Sigma g_1(n)x^n$, which reduces to $xg_1(1)$, would be regular at ∞, which is impossible. Hence $x = 1$ is actually a singular point.

We remark that unless g is a polynomial, the point 1 is an essential singularity. For if this point is a pole of order k, there will exist a polynomial $P_{k-1}(z)$ such that $a_n = P_{k-1}(n)$, $n = 1, 2, \cdots$. Then $g(n) - P_{k-1}(n) = 0$, $n = 1, 2, \cdots$. But the function $g_2(z) = g(z) - P_{k-1}(z)$ satisfies the hypothesis of the theorem. Hence, as we have just seen, it cannot vanish at the points $1, 2, \cdots$.

Returning now to the proof of the main part of the theorem, we consider the functions

$$f_0(x) = \frac{x}{1-x} = \sum_{m=1}^{\infty} x^m,$$

$$f_1(x) = H_x^{(1)} f_0(x) = \sum_{m=1}^{\infty} m x^m,$$

. . . . . .

$$f_n(x) = H_x^{(n)} f_0(x) = \sum_{m=1}^{\infty} m^n x^m.$$

We shall prove that for each n there exist n positive numbers $b_p^{(1)}, b_p^{(2)}, \cdots, b_p^{(n)}$ such that

$$f_n(x) = \sum_{p=1}^{n} x^p b_p^{(n)} f_0^{(p)}(x), \tag{1}$$

and

$$b_n^{(n)} = 1, \; b_p^{(n)} \leq \frac{2^n (n-1)!}{p!}, \; p \leq n. \tag{2}$$

Suppose (1) is true for n. We have

$$f_{n+1}(x) = x f'_n(x),$$

$$f'_n(x) = \sum_{p=1}^{n} b_p^{(n)} [p x^{p-1} f_0^{(p)}(x) + x^p f_0^{(p+1)}(x)].$$

Hence

$$f_{n+1}(x) = \sum_{p=1}^{n} b_p^{(n)} [p x^p f_0^{(p)}(x) + x^{p+1} f_0^{(p+1)}(x)],$$

and if for convenience we write $b_0^{(n)} = 0$, we have the recurrence formulas

$$b_p^{(n+1)} = b_p^{(n)} p + b_{p-1}^{(n)}, \; p = 1, 2, \cdots, n,$$

$$b_{n+1}^{(n+1)} = 1.$$

Therefore (1) is true for $n + 1$.

If (2) is valid for n, we have

$$b_p^{(n+1)} \leq \frac{2^n(n-1)!}{p!}p + \frac{2^n(n-1)!}{(p-1)!}, \quad p \leq n+1$$

$$= \frac{2^{n+1}(n-1)!}{(p-1)!}$$

$$\leq \frac{2^{n+1}n!}{p!}.$$

Hence (2) holds for $n+1$.

Each $f_n(x)$ has the point $x = 1$ as its only singularity. Let us determine an upper bound for these functions in the region exterior to a circle with center at $x = 1$ and of arbitrary radius.

$$\left| x^p f_0^{(p)}(x) \right| = \left| x^p \frac{d^p}{dx^p} \frac{x}{1-x} \right| < p! \frac{|x|^p}{|1-x|^{p+1}}.$$

$$\left| f_n(x) \right| = \left| \sum_{p=1}^{n} x^p b_p^{(n)} f_0^{(p)}(x) \right|$$

$$< 2^n(n-1)! \sum_{p=1}^{n} \frac{|x|^p}{|1-x|^{p+1}}.$$

For x outside the circle, suppose $\left|1-x\right| > h$, $\left|\frac{x}{1-x}\right| < M$, a constant greater than 1. Then

$$\frac{|x|^p}{|1-x|^{p+1}} < \frac{M^n}{|1-x|},$$

$$\sum_{p=1}^{n} \frac{|x|^p}{|1-x|^{p+1}} < \frac{nM^n}{|1-x|},$$

$$2^n(n-1)! \sum_{p=1}^{n} \frac{|x|^p}{|1-x|^{p+1}} < \frac{2^n n! M^n}{|1-x|},$$

$$|f_n(x)| < n! H^n,$$

where H is a constant.

It was proved by Poincaré that if $g(z) = \Sigma\alpha_n z^n$ is an integral function which satisfies the hypothesis of this theorem, then $\Sigma n!\,\alpha_n z^n$ is also an integral function. We have $|f_n(x)| < n!\,H^n$, and since $\Sigma n!|\alpha_n|\,|z|^n$ converges for all z, the series $\Sigma n!\,|\alpha_n|\,H^n$ converges. Hence $\Sigma\,|\alpha_n f_n(x)|$ converges, and the convergence is uniform outside the circle. In this region, the function

$$F(x) = \sum_{n=0}^{\infty} \alpha_n f_n(x)$$

is accordingly holomorphic.

Consider the sequence $\{S_m(x)\}$,

$$S_m(x) = \alpha_0 f_0(x) + \cdots + \alpha_m f_m(x).$$

We have seen that $\lim_{m\to\infty} S_m(x)$ is a function having its only possible singularity at $x = 1$, since $S_m(x)$ converges uniformly to $\Sigma\alpha_n f_n(x)$ in every region exterior to a circle with center at $x = 1$ and of radius arbitrarily small. Hence the sequence $S_m^{(k)}(x)$ converges uniformly in the same region to $\frac{d^k}{dx^k}\sum_{n=0}^{\infty} \alpha_n f_n(x)$. We have

$$\frac{d^k}{dx^k}\,\alpha_m f_m(x)\,|_{x=0} = \alpha_m k^m k!,$$

$$S_m^{(k)}(0) = k!\sum_{i=0}^{m} \alpha_i k^i.$$

Hence $\lim S_m^{(k)}(x)\,|_{x=0} = k!\,g(k)$ is the k-th derivative of $\sum_{n=0}^{\infty} \alpha_n f_n(x)$ at $x = 0$. The function $\sum_{n=0}^{\infty} \alpha_n f_n(x)$, being regular for $|x| < 1$, may be developed in a Taylor's series, and the coefficient of order k is therefore $g(k)$. Hence we have

$$f(x) = f(0) + \sum_{n=1}^{\infty} g(n)x^n$$

$$= F(x) + f(0) - g(0),$$

and therefore $f(x)$ has the point $x = 1$ as its only possible singularity. Thus the sufficiency of the condition is proved.

To prove the condition necessary, we note first that since $x = 1$ is the only singularity, $f(x)$ may be regarded as an integral function of the variable $\xi = \dfrac{1}{1-x}$, and therefore as an integral function of $\xi - 1 = \dfrac{x}{1-x}$:

$$f(x) = \sum_{n=0}^{\infty} A_n \left(\frac{x}{1-x}\right)^n, \quad \lim_{n\to\infty} \sqrt[n]{|A_n|} = 0.$$

Expand each term in a Taylor's series:

$$\left(\frac{x}{1-x}\right)^n = \sum_{k=0}^{\infty} C_{n+k-1}^{k} x^{n+k}, \quad |x| < 1$$

$$= \sum_{m=n}^{\infty} C_{m-1}^{m-n} x^m,$$

where, for convenience, $C_{m-1}^{m} = 0$, $m > 0$, and $C_{-1}^{0} = 1$. Consequently

$$f(x) = \sum_{n=0}^{\infty} A_n \sum_{m=n}^{\infty} C_{m-1}^{m-n} x^m$$

$$= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} A_n \, C_{m-1}^{m-n} \right) x^m.$$

Hence we have

$$a_m = \sum_{n=0}^{m} A_n C_{m-1}^{m-n},$$

and, for $m \geq 1$,

$$a_m = A_1 + A_2(m-1) + A_3 \frac{(m-1)(m-2)}{2!} + \cdots + A_m. \quad (2)$$

We take now as the function $g(z)$ the following:

$$g(z) = A_1 + A_2(z-1) + \cdots + A_m \frac{(z-1)(z-2)\cdots(z-m+1)}{(m-1)!} + R_m(z), \quad (3)$$

and show first that it is an integral function. Let r be an arbitrary positive number. We must show that the series for $g(z)$ converges uniformly in the circle of radius r. Since $\lim_{n\to\infty} \sqrt[n]{|A_n|} = 0$, we have

$$|A_m| < \epsilon^m \text{ for } m > m_0.$$

Hence, for $m > m_0$,

$$|R_m(z)| < \epsilon \left[\frac{(r+1)\cdots(r+m)}{m!}\epsilon^m + \frac{(r+1)\cdots(r+m+1)}{(m+1)!}\epsilon^{m+1} + \cdots \right].$$

This expression is the remainder after $m+1$ terms of the development of $\frac{1}{(1-\epsilon)^{r+1}}$; hence its limit as m becomes infinite is zero, since ϵ may be taken less than 1. The series (3) accordingly represents an integral function.

Moreover, for any ϵ, $0 < \epsilon < 1$, $|g(z)| < e^{r\epsilon}$ for $r > r_\epsilon$. For

$$\log|R_m(z)| < \log\epsilon + \log\left[\frac{(r+1)\cdots(r+m)}{m!}\epsilon^m + \frac{(r+1)\cdots(r+m+1)}{(m+1)!}\epsilon^{m+1} + \cdots \right]$$

$$< \log\epsilon + \log\frac{1}{(1-\epsilon)^{r+1}}$$

$$< \log\frac{1}{(1-\epsilon)^{r+1}}.$$

$$|R_m(z)| < e^{-(r+1)\log(1-\epsilon)},$$

$$|g(z)| < |P_m(z)| + e^{-(r+1)\log(1-\epsilon)},$$

where $P_m(z)$ is a polynomial of degree m.

Now corresponding to an arbitrary $\eta > 0$, the number ϵ may be taken so small that $|\log(1-\epsilon)| < \frac{\eta}{3}$. Then

$$|g(z)| < |P_m(z)| + e^{\frac{\eta}{3}(r+1)}$$

But $|P_m(z)| < Mr^m$ for all r. Moreover, $Mr^m < e^{\frac{r\eta}{3}}$ for r sufficiently large. Hence

$$|g(z)| < e^{\frac{\eta r}{3}} + e^{\frac{\eta r}{3}} e^{\frac{\eta}{3}}$$
$$< e^{\eta r}$$

From (2) and (3) we have $g(n) = a_n$. The proof is therefore complete.[1]

41. The following theorem, whose proof is omitted, is due to Le Roy and Lindelöf:

THEOREM 2: *Let $g(z)$ be holomorphic in a certain half plane. If we have, for $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$ and $r > r_\epsilon$,*

$$|g(\alpha + re^{\psi i})| < e^{(\theta+\epsilon)r},$$

where $\theta_0 < \pi$ is a constant, and $z - \alpha = re^{\psi i}$, then $f(x) = \Sigma g(n)x^n$ is holomorphic for all x in the sector $\theta_0 < \theta < 2\pi - \theta_0$, where $x = \rho e^{i\theta}$.

In particular the conclusion is verified if $g(z)$ is an integral function such that

$$|g(z)| < e^{(\theta_0+\epsilon)r} \text{ for } r > r_\epsilon, \tag{5}$$

the inequality being true for all ψ. The converse of this statement (with a hypothesis somewhat more general) was proved by the author for the case in which $f(x)$ has on the circle of convergence only a finite number of singularities on the arc $\theta_0 < \theta < 2\pi - \theta_0$. Then $f(x)$ may be represented as $\Sigma a_n x^n$ where $a_n = g(n)$, and $g(n)$ satisfies the inequality (5).

[1] The fact that two integral functions $g_1(z)$, $g_2(z)$, with $|g_i(z)| < e^{\epsilon r}$, can not be found so as to be the same at all of the values $z = 0, 1, 2, \ldots$ except a finite number (see § 42) implies that the $g(z)$ of the theorem can not in general take on the value a_0 when $z = 0$. [EDITOR.]

IX

FATOU'S THEOREM

42. In the theorems of the preceding chapters, the hypotheses, insofar as they have involved the coefficients in a Taylor's series, have referred mainly to their absolute values. The object of this chapter is to show that the algebraic signs of the a_n affect the number and the character of the singularities. Theorems of this kind originated in an important theorem of Fatou.[1]

THEOREM 1: *Given the series*

$$\sum a_n x^n, \quad 0 < R < \infty, \tag{1}$$

there exists a sequence $\{\lambda_n\}$ *such that the series obtained by changing the signs of the* a_{λ_n} *has the circle of convergence as a cut.*

Write $a_n = \alpha_n + i\beta_n$. Fatou's proof of the theorem required that $\lim \alpha_n = 0$, $\lim \beta_n = 0$, and $\Sigma\sqrt{\alpha_n^2 + \beta_n^2}$ diverges. He remarked, however, that it is probable that the conclusion is valid without any restriction on the $\Sigma\sqrt{a_n^2 + b_n^2}$. That this is indeed the case was proved by Hurwitz and Polya. It is a modification of Hurwitz's proof that we give here, not only because of its elegance, but also because the method can be applied to the proof of other theorems, in particular to Theorem 2 below.

[1] Acta Math., t. 30 (1906), p. 335. See also Hurwitz and Polya, Acta Math., t. 40 (1916), p. 179, and Mandelbrojt, Ann. de l'École Norm. Sup., t. 40 (1923), p. 413.

Assume $R = 1$. If the circle of convergence is not a cut there will be at least one rational [1] value of ϕ for which $f(e^{i\phi})$ is holomorphic.

Select a sequence $\{n_i\}$ for which

$$\lim_{i\to\infty} (n_{i+1} - n_i) = \infty, \tag{2}$$

$$\lim_{i\to\infty} \sqrt[n_i]{|a_{n_i}|} = 1, \tag{3}$$

and such that none of the a_i vanishes. Let

$$Q(x) = \sum_{i=1}^{\infty} a_{n_i} x^{n_i},$$

$$P(x) = \sum_{n=0}^{\infty} a_n x^n - Q(x)$$

$$= \sum_{j=1}^{\infty} a_{n'_j} x^{n'_j}.$$

Arrange the sequence $\{a_{n_i}\}$ in the following form:

$$\begin{array}{llll} a_{n_1} & & & \\ a_{n_2} & a_{n_3} & & \\ a_{n_4} & a_{n_5} & a_{n_6} & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}$$

Denote by A_k the set of elements in the k-th column, and by $(-1)A_k$ the set obtained by changing the sign of each element of A_k. Consider the sets

$$(-1)^{\alpha_1} A_1,\ (-1)^{\alpha_2} A_2, \cdots, \tag{4}$$

where $\alpha_1, \alpha_2, \cdots$ take on the values 0 and 1. The totality of these sets has the power of the continuum. Let $\eta = 0.\alpha_1\alpha_2\cdots$, $0 \le \eta \le 1$. Denote the corresponding set (4) by

$$(-1)^{\alpha_1^{(\eta)}} A_1,\ (-1)^{\alpha_2^{(\eta)}} A_2, \cdots \tag{4'}$$

[1] Hurwitz considers, instead of rational points, arcs on which $f(e^{i\phi})$ is holomorphic.

If a number η can be written in two different ways, we shall retain in (4) only one of them. The set will still have the power of the continuum. Let $Q_\eta(x) = \Sigma c_{n_i}^{(\eta)} x^{n_i}$, where the $c_{n_i}^{(\eta)}$ are the elements of the set (4′), and let $f_\eta(x) = Q_\eta(x) + P(x)$.

Suppose that for each η, none of the $f_\eta(x)$ has the circle of convergence as a cut. Then, since (*a*), each $f_\eta(x)$ has at least one regular point $e^{i\phi_\eta}$, where ϕ_η is rational, and the set of such ϕ_η accordingly denumerable, and (*b*), the $f_\eta(x)$ constitute a set having the power of the continuum, there will be at least two different numbers η_1, η_2 such that $f_{\eta_1}(x)$ and $f_{\eta_2}(x)$ are regular at the same point $e^{i\phi} = e^{i\phi_{\eta_1}} = e^{i\phi_{\eta_2}}$. Then the function

$$f_{\eta_1}(x) - f_{\eta_2}(x) = Q_{\eta_1}(x) - Q_{\eta_2}(x)$$

is regular at that point. This, however, is impossible. In fact, since η_1 and η_2 are distinct, there will be some k such that if the α_k of η_1 is 0, the α_k of η_2 is 1, or *vice versa*. Denote by $c_{n_{i_l}}^{(\eta_1)}$ the $c_{n_i}^{(\eta_1)}$ belonging to the partial set $(-1)^{\alpha_k^{(\eta_1)}} A_k$ which differ in sign from the $c_{n_i}^{(\eta_2)}$ belonging evidently to $(-1)^{\alpha_k^{(\eta_2)}} A_k$. Then

$$Q_{\eta_1}(x) - Q_{\eta_2}(x) = \sum d_{n_i} x^{n_i}, \tag{5}$$

where at least the coefficients $d_{n_{i_l}} = 2\, c_{n_{i_l}}^{(\eta_1)}$ are not zero. Hence

$$\left| d_{n_{i_l}} \right| = 2 \left| a_{n_{i_l}} \right| . \tag{6}$$

The series for $Q_{\eta_1}(x)$ and $Q_{\eta_2}(x)$ both have unit radius of convergence. Hence the radius of convergence r of (5) is at least 1. Consequently, from (3) and (6), $r = 1$. From the way in which the n_i were chosen, the series (5) has its circle of convergence as a cut. Hence the assumption that no $f_\eta(x)$ has its circle of convergence as a cut leads to a contradiction. But any $f_\eta(x)$ for which the circle of conver-

gence is a cut has the same coefficients as the given function except for changes in sign of an infinity of coefficients. The theorem is therefore proved.

43. THEOREM 2:[1] *Given* $f(x) = \Sigma a_n x^n$, *there exists a sequence* $\{n_i\}$ *such that it suffices to change the signs of the coefficients* a_{n_i} *in order that the circle of convergence of the resulting series is a cut, and every point on the circle of convergence has the maximum order, viz., the order* ω *of* $f(x)$ *on the circle of convergence:*

$$\omega = \overline{\lim_{n\to\infty}} \frac{\log |a_n|}{\log n} + 1.$$

If a function represented by a series with unit radius of convergence has on the circle of convergence a singularity of order $\omega_1 < \omega$, then, by definition, this point may be enclosed in an arc in which the order of the function is less than ω. It follows that if the order of the function is $\omega' < \omega$ in $e^{i\phi}$, there is a rational number ϕ_1 such that the order in $e^{i\phi_1}$ is less than ω. If two functions have in the point x_0 the orders ω', ω'', $< \omega$, respectively, the sum of the two functions has in x_0 the order $\omega''' < \omega$, where ω''' is the greater of ω' and ω''.

The proof of Theorem 2 is similar to that of Theorem 1. We now choose the sequence $\{n_i\}$ so that

$$n_{i+1} - n_i > \sqrt{n_i \log n_i} \text{ for } i > i_0, \text{ and also}$$

$$\lim_{i\to\infty} \frac{\log |a_{n_i}|}{\log n_i} + 1 = \omega = \overline{\lim_{n\to\infty}} \frac{\log |a_n|}{\log n} + 1.$$

Instead of Hadamard's theorem, we apply Fabry's theorem [2] on the order of singularities. Replace "regular point" by "point of order $\omega' < \omega$," and the proof is the same as that of Fatou's theorem.

[1] Mandelbrojt, Comptes Rendus, t. 184 (1927), p. 509.
[2] Chapter VII, Theorem 8.

44. In conclusion, we state the following theorem:

THEOREM 3 [1]: *Given a sequence* $\{n_i\}$ *such that*

$$\lim_{i\to\infty}(n_{i+1}-n_i)=\infty,\quad \lim_{i\to\infty}\sqrt[n_i]{|a_{n_i}|}=1.$$

Denote by

$$\sum a_n^{(\phi)}x^n,\quad 0\leq\phi\leq 2\pi, \tag{7}$$

all series such that

$$a_{n_i}^{(\phi)}=e^{i\phi}a_{n_i},$$

$$a_n^{(\phi)}=a_n,\quad n\neq n_i.$$

There exists a set of values of ϕ *having the power of the continuum such that the corresponding series* (7) *has the circle of convergence as a cut, and a denumerable set of values of* ϕ *for which the corresponding series does not have the circle of convergence as a cut.*

[1] Mandelbrojt, Ann. de l'École Norm. Sup., t. 40 (1923), p. 413.

X

COMPLEMENTS AND APPLICATIONS OF THE THEOREMS OF HADAMARD AND HURWITZ

45. We have seen that if $f(x)$, $\phi(x)$ have isolated singularities α, β, respectively, the functions remaining uniform about these points, we can write

$$f(x) = \sum_{i=1}^{n} f_i(x) + F_n(x),$$

$f_i(x)$ having only α_i as a singularity, and $F_n(x)$ having no singularities within a circle C_R of arbitrary radius R, if n is sufficiently large. Similarly

$$\phi(x) = \sum_{i=1}^{m} \phi_i(x) + \Phi_m(x),$$

with a corresponding statement for C_{R_1}. We have also seen that

$$H(f, \phi) = \sum_{i,j=1,1}^{n,m} H(f_i, \phi_j) + \sum_{j=1}^{m} H(F_n, \phi_j) + \sum_{i=1}^{n} H(\Phi_m, f_i). \tag{1}$$

Denote by r, r_1 the radii of convergence of the series for f, ϕ, respectively. None of the series

$$\sum_{j=1}^{m} H(F_n, \phi_j)$$

has a singularity in the circle of radius Rr_1, and none of the series

$$\sum_{i=1}^{n} H(\Phi_m, f_i)$$

has a singularity in the circle of radius $R_1 r$. Let R' be arbitrarily large. Since R is arbitrary, there exists for each R' a number R such that all the singularities of $H(f,\phi)$ in the circle of radius R' occur in

$$\sum_{i,j=1,1}^{n,m} H(f_i, \phi_j).$$

In fact we need only take R, R_1 so that Rr_1 and $R_1 r$ exceed R'. Then in the circle $C_{R'}$ are singularities only of the form $\gamma_{ij} = \alpha_i \beta_j$. On the assumption that γ_{ij} can be obtained in only one manner, it will be a singularity of only one function $H(f_i, \phi_j)$. The remaining functions in (1) will be regular in γ_{ij}. That is, $H(f_i, \phi_j)$ is the principal part of the singularity of $H(f, \phi)$ in γ_{ij}. But f_i, ϕ_j are the principal parts of the singularities of f in α_i and ϕ in β_j, respectively. Hence Borel's remark that the kind of singularity of $H(f,\phi)$ in $\gamma_{ij} = \alpha_i \beta_j$ depends only on the kind of singularity of f in α_i and that of ϕ in β_j. We have verified the statement for isolated non-critical points; the remark applies, however, in certain cases to singularities of other kinds. With Borel, we make precise this remark in the following theorem:

46. Theorem 1: *If γ_{ij} is obtainable in only one way, and if α_i is a pole of order q of f, and β_j is a pole of order p of ϕ, then γ_{ij} is necessarily a singularity of $H(f,\phi)$, and is a pole of order $p + q - 1$.*

Because of the preceding remarks, it suffices to give the proof for the case in which the point 1 is a pole of order q of f, and of order p of ϕ, the functions f and ϕ being regular elsewhere. In order to justify the assumption that the poles are situated at the point 1, suppose α_0, β_0 are the respective poles of f and ϕ. Then the series

$$\sum a_n \alpha_0^n x^n, \quad \sum b_n \beta_0^n x^n$$

have poles at the point 1. If we can show that the series

$$\sum a_n b_n (\alpha_0 \beta_0)^n x^n$$

has a pole of order $p + q - 1$ at the point 1, it will follow that

$$\sum a_n b_n x^n$$

has a pole of order $p + q - 1$ at the point $\alpha_0 \beta_0$.

We may write

$$f(x) = A_0 + \frac{A_1}{1-x} + \cdots + \frac{A_q}{(1-x)^q},$$

$$\phi(x) = B_0 + \frac{B_1}{1-x} + \cdots + \frac{B_p}{(1-x)^p}.$$

Now

$$\frac{A_k}{(1-x)^k} = \sum_{n=0}^{\infty} \frac{A_k}{(k-1)!}(n+k-1)(n+k-2)\cdots(n+1)x^n$$

$$= \sum Q_{k-1}(n)x^n$$

where $Q_0(n) = A_0$, $Q_{k-1}(n) = \frac{A_k}{(k-1)!}(n+k-1)\cdots(n+1)$.

Then $f(x) = \sum U_{q-1}(n)x^n$, and similarly, $\phi(x) = \sum V_{p-1}(n)x^n$,

writing $U_{q-1}(n) = \sum_{k=1}^{q} Q_{k-1}(n)$, $V_{p-1} = \sum_{k=1}^{p} P_{k-1}(n)$.

Hence, $H(f, \phi) = \sum R_{p+q-2}(n)x^n$,

where $R_{p+q-2} = U_{q-1} V_{p-1}$

Let $R_{p+q-2} = c_0 + c_1 n + \cdots + c_{p+q-2}(n)^{p+q-2}$

Then $\sum R_{p+q-2}(n)x^n = \sum (c_0 + \cdots + c_{p+q-2}n^{p+q-2})x^n$

$$= c_0 \sum x^n + c_1 \sum n x^n + \cdots + c_{p+q-2} \sum n^{p+q-2} x^n$$

Each of the functions in these series $\Sigma n^k x^n$ has a pole of order $k+1$ in the point 1. Consequently $H(f,\phi)$ has in the point 1 a pole of order $p+q-1$.

47. THEOREM 2: *If γ_{ij} is determined in only one way, and if α_i, β_j are essential singularities of f,ϕ, respectively, then γ_{ij} is an essential singularity of $H(f,\phi)$.*

Assume that the point 1 is an essential singularity of f and ϕ, and that these functions have no other singularities. If

$$f(x) = \sum a_n x^n, \quad \phi(x) = \sum b_n x^n,$$

then, from Faber's theorem, $a_n = g(n)$, where $g(n)$ is an integral function such that

$$|\, g(z)\,| < e^{\epsilon r} \text{ for } r > r_\epsilon, \quad r = |\, z\,|.$$

Similarly $b_n = g_1(n)$, where $|\, g_1(z)\,| < e^{\epsilon r}$ for $r > r'_\epsilon$. Hence $G = gg_1$, an integral function (not a polynomial, since g, g_1 are not polynomials) such that

$$|\, G(z)\,| < e^{\epsilon r}, \quad r > r''_\epsilon,$$

r''_ϵ being the greater of $r_{\frac{\epsilon}{2}}$ and $r'_{\frac{\epsilon}{2}}$.

By Faber's theorem, the series

$$\sum G(n)x^n = \sum a_n b_n x^n = H(f,\phi)$$

has the point 1 as its only singularity, and from the remarks in § 42 it follows that this point is surely singular for H, and likewise essential.

It is to be noted that the point 1 is an essential singularity of $H(f,\phi)$ if this point is an essential singularity merely of one of the functions f, ϕ, provided that for the other function it is an isolated, non-critical singular point.[1]

In general, if the α, β are not isolated, it may happen, as shown by an example of Soula,[2] that even if γ_{ij} is determined

[1] Faber, Jahresb. der deutsche Math. Verein., bd. xvi (1907), p. 285.

[2] Jour. de Math. pures et appliquées, t. 86 (1921), p. 197.

in only one way, none of the points γ is a singularity of $H(f,\phi)$. Another example is the following. Suppose

$$f(x) = \sum a_{\lambda_n} x^{\lambda_n}, \quad \overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = \infty,$$

the series having the point 1 as its only singularity. The existence of such a series was proved in Chapter V. Let $\{\lambda_n'\}$ denote a sequence chosen from the sequence complementary to $\{\lambda_n\}$ and having the property that

$$\lim_{n\to\infty} (\lambda_{n+1}' - \lambda_n') = \infty.$$

Form the function

$$\phi(x) = \sum_{n=0}^{\infty} b_{\lambda'_n} x^{\lambda'_n},$$

the series having unit radius of convergence. Then $H(f,\phi) = 0$. By Hadamard's theorem, the series for $\phi(x)$ has its circle of convergence as a cut. Its singularities therefore consist of all points whose absolute value is at least 1. Since $\alpha = 1$, we have $|\alpha\beta| = |\beta| \geq 1$. But none of the points $\alpha\beta$ is a singularity of $H(f,\phi)$ which vanishes identically.

48. We now show that the theorem of Hurwitz may be applied to give the angle of at least one singular point on the circle of convergence.

THEOREM 3 (Mandelbrojt): *Let $g(z)$ be an integral function such that*

$$|g(z)| < e^{r^r} \text{ for } r > r_0,\ r = |z|.$$

Let $f(x) = \Sigma a_n x^n$, a series with unit radius of convergence, and having all its singularities on the circle of convergence. Let

$$\gamma_n = a_1 g(n+1) - C_n^1 a_2 g(n) + \cdots + (-1)^n a_{n+1} g \quad (1),$$

$$k = \overline{\lim} \sqrt[n]{|\gamma_n|}.$$

Then one of the angles $\cos^{-1}\left(1 - \frac{k^2}{2}\right)$ is the argument of at least one singularity of $f(x)$.

Proof: Consider the series

$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{z^{n+1}}, \tag{2}$$

and

$$\sum_{n=0}^{\infty'} \frac{(-1)^n g(n+1)}{z^{n+1}}. \tag{3}$$

Denote by α the singularities of $f(x)$. Then $|\alpha| = 1$. The series (2) has singularities only of the form $\alpha' = \frac{1}{\alpha}$. The series (3) has the point -1 as its only singularity, since, by Faber's theorem, the point 1 is the only singularity of $\Sigma g(n+1)x^{n+1}$. The operation of Hurwitz applied to (2) and (3) gives

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{z^{n+1}}. \tag{4}$$

Every singularity γ of (4) is of the form $\alpha' - 1$. The radius of convergence of (4) is given by

$$k = \overline{\lim} \sqrt[n]{|\gamma_n|}.$$

Hence there exists a singularity of (4), say $ke^{i\phi_0}$, $0 \leq \psi_0 \leq 2\pi$, and α_0' such that

$$\begin{aligned} ke^{i\psi_0} &= \alpha_0' - 1 \\ &= e^{i\phi_0} - 1, \quad \alpha_0' = e^{i\phi_0}. \end{aligned}$$

That is,

$$k(\cos \psi_0 + i \sin \psi_0) = \cos \phi_0 - 1 + i \sin \phi_0.$$

Equating the real and the imaginary parts, we obtain

$$\cos \phi_0 = 1 - \frac{k^2}{2}.$$

At least one of the ϕ_0 thus determined is the angle of a singularity α' of (2). But since $\alpha = \frac{1}{\alpha'}$, the same is true for $f(x)$.

By means of this theorem it is possible to determine the positions of the singularities of certain functions all of whose singularities are on the circle of convergence.

XI

SINGULARITIES AND ARITHMETICAL PROPERTIES OF THE COEFFICIENTS

49. At the beginning of Chapter V, we remarked that some general facts about the singularities of a function may be inferred from certain arithmetical properties of the coefficients, particularly such properties as remain invariant with respect, for example, to differentiation. In that chapter the property considered was the number and the distribution of the lacunae.

We proceed now to a group of theorems which show that with a knowledge of certain other arithmetical properties of the coefficients we may predict the form of the function. The theorems of the present chapter, while therefore related to earlier theorems, differ from those theorems in that the hypotheses on the coefficients are more specialized, and accordingly the determination of the nature of the function is more specific.

50. THEOREM 1 (Eisenstein [1]): *A series*

$$\sum_{n=0}^{\infty} A_n x^n, \quad R = 1, \tag{1}$$

where the A_n are rational, has the property that if the function defined by the series is algebraic, there will exist an integer N such that the quantities $A_n N^n$ are integers.

The converse is not true, for we have seen that in any series with a finite radius of convergence it suffices to change the

[1] Monatshefte d. Akad. d. Wiss. zu Berlin (1852), p. 441.

signs of an infinity of coefficients in order that the circle of convergence of the resulting series shall be a cut. Consequently the function represented by the altered series, although its coefficients have the required property, can not be algebraic.

For the case in which all the A_n are integers, we have the theorem of Carlson [1]:

THEOREM 2: *If the circle of convergence of* (1) *is not a cut, the series represents a rational function of the form*

$$\frac{P(x)}{(1-x^p)^q},$$

where $P(x)$ *is a polynomial, and* p, q *are positive integers.*

51. The author [2] has proved a theorem which may in certain cases be regarded as the converse of the theorem of Eisenstein.

THEOREM 3: *If the series* (1) *has rational coefficients such that there exists an integer* N *for which the quantities*

$$A_0,\ A_1N,\cdots,\ A_nN^n,\cdots$$

are integers, and if the function represented by the series is regular exterior to and on the circumference of the circle of radius $\frac{N}{N^2-1}$ *and center* $\frac{N^2}{N^2-1}$, *then the function is of the form*

$$\frac{P(x)}{(1-x)^h},$$

where $P(x)$ *is a polynomial, and* h *is an integer.*

We remark that if the above condition on the distribution of the singularities is dispensed with, the function defined

[1] Math. Zeitschrift, bd. 9 (1921), p. 1.
[2] Comptes Rendus, t. 178 (1924), p. 985.

by (1) may have an almost arbitrary distribution of singularities. The theorem is therefore of a character quite different from that of Carlson.

For the proof of Theorem 3, let

$$\left.\begin{aligned}\phi(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n N^n}{z^{n+1}},\\ f(z) &= \sum_{n=0}^{\infty} \frac{A_{n+1} N^{n+1}}{z^{n+1}}.\end{aligned}\right\} \tag{2}$$

The singularities of $f(z)$ are the points which correspond by the transformation

$$z = \frac{N}{x} \tag{3}$$

to the singularities of $\sum_{n=1}^{\infty} A_n x^n$. A direct calculation, on the basis of (3), shows that to the circle C with center at $\frac{N^2}{N^2-1}$ and radius $\frac{N}{N^2-1}$ corresponds in the z-plane the circle C_1 with center at N and radius 1. For the transformation carries real values of x into real values of z; the circle C has its center on the axis of reals, and intersects this axis at the points

$$x_1 = \frac{N^2}{N^2-1} + \frac{N}{N^2-1} = \frac{N}{N-1},$$

$$x_2 = \frac{N^2}{N^2-1} - \frac{N}{N^2-1} = \frac{N}{N+1},$$

to which correspond the real points

$$z_1 = N-1, \quad z_2 = N+1.$$

Hence C_1 has its center at $\frac{1}{2}(z_1 + z_2) = N$, and radius 1.

The theorem of Hurwitz applied to the series (2) shows that the function

$$\psi(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{z^{n+1}},$$

where

$$\gamma_n = (A_1 N)N^n - C_n^1(A_2 N^2)N^{n-1} + \cdots \pm (A_n N^n)N$$
$$= \delta\, N^{n+1}\Delta_n A_1, \tag{4}$$

$\Delta_n A_1$ denoting the nth difference of the sequence $\{A_n\}$ with respect to A_1, and δ being equal to 1 or -1, is regular exterior to and on the circumference of the unit circle, since the only singularity of $\phi(z)$ is the point $-N$. Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\gamma_n|} = \theta < 1.$$

But from the hypothesis and (4), the γ_n are integers. Hence $\gamma_n = 0$ for $n > n_0$.

In the series $\sum_{k=1}^{\infty} A_k x^{k-1}$, replace x by $\frac{y}{1+y}$. We have

$$\sum_{k=1}^{\infty} A_k \left(\frac{y}{1+y}\right)^{k-1} = (1+y)\sum_{k=1}^{\infty} A_k \frac{y^{k-1}}{(1+y)^k}.$$

The general term of this series may be written

$$A_k \frac{y^{k-1}}{(1+y)^k} = \sum_{r=k-1}^{\infty} (-1)^{r-k+1} A_k C_r^{k-1} y^r.$$

Hence

$$\sum_{k=1}^{\infty} A_k \frac{y^{k-1}}{(1+y)^k} = \sum_{k=1}^{\infty}\sum_{r=k-1}^{\infty} A_k C_r^{k-1} y^r (-1)^{r-k+1}$$
$$= \sum_{r=0}^{\infty} y^r \sum_{k=1}^{r+1} A_k C_r^{k-1}(-1)^{r-k+1}.$$

Then

$$\sum_{k=1}^{\infty} A_k x^{k-1} = \frac{1}{1-x} \sum_{r=0}^{\infty} \left(\frac{x}{1-x}\right)^r (-1)^r \sum_{k=1}^{r+1} (-1)^{k-1} C_r^{k-1} A_k$$

$$= \sum_{r=0}^{\infty} \left(\frac{x}{1-x}\right)^r \frac{1}{1-x} (-1)^r \Delta_r A_1$$

$$= \sum_{r=0}^{n_0} \left(\frac{x}{1-x}\right)^r \frac{1}{1-x} (-1)^r \Delta_r A_1$$

$$= \frac{P_1(x)}{(1-x)^h},$$

so that

$$A_0 + A_1 x + A_2 x^2 + \cdots = A_0 + \frac{x P_1(x)}{(1-x)^h}$$

$$= \frac{P(x)}{(1-x)^h},$$

and the theorem is proved.

Achyeser [1] has shown that the preceding theorem may be proved by means of Euler's transformation.

We remark an interesting consequence of this theorem. Given the series (1). If N is a positive integer such that

$$\Sigma A_n N^n x^n$$

has integral coefficients, then unless $\Sigma A_n x^n$ represents a function of the form $\frac{P(x)}{(1-x)^h}$ we must have

$$N \geq \frac{1}{\rho} - 1, \tag{5}$$

where ρ is the greatest distance from the point 1 to a singularity.

[1] Bull. de l'Acad. des Sciences de l'Oucraïne, t. 1 (1925), p. 32.

We may suppose that ρ is less than 1, for otherwise the statement is trivial. Write (5) in the form

$$\frac{1}{N+1} \leq \rho.$$

If this inequality were not true, all the singularities would lie within the circle of center $\frac{N^2}{N^2-1}$ and radius $\frac{N}{N^2-1}$. The function defined by $\Sigma A_n x^n$ would therefore be of the form $\frac{P(x)}{(1-x)^h}$, with only one singularity.

We thus obtain a lower bound for N in order that a function represented by a series with rational coefficients be, for instance, algebraic, not rational.

The author has applied the foregoing type of research to the theory of transcendental numbers.

XII

SERIES HAVING THE CIRCLE OF CONVERGENCE AS A CUT

Theorem 1 of Chapter V is sometimes called the Hadamard-Fabry theorem. This chapter has for its object the consideration of related theorems. We shall obtain less restrictive conditions on the $\{\lambda_n\}$ in order that the circle of convergence shall be a cut. We begin by developing some ideas which, although elementary, are important in themselves as well as in their bearing on the theorems in question.

Let $f(x)$ denote the function defined by $\Sigma a_n x^n$ within the unit circle. If $|\beta| < 1$, we have

$$f(\beta) = a_0 + a_1\beta + \cdots,$$

$$f^{(m)}(\beta) = \sum_{q=0}^{\infty} \frac{(m+q)!}{q!} a_{m+q}\beta^q.$$

For the expansion of $f(x)$ about the point $x = \beta$, we may write

$$f(x) = \sum_{m=0}^{\infty} c_m(x-\beta)^m, \quad \beta = be^{i\phi}.$$

Let C_1, R_1 denote, respectively, the circle and the radius of convergence of this series. We have

$$c_m = \frac{f^{(m)}(\beta)}{m!} = \sum_{q=0}^{\infty} \frac{(m+q)!}{m!\,q!} a_{m+q}\beta^q. \tag{1}$$

It is clear that $R_1 \geq 1 - b$, since $f(x)$ has no singularities within the circle C_2, center at β, radius $R_2 = 1 - b$. Hence

C_2 is tangent to the unit circle at the point $e^{i\phi}$. If $R_1 = R_2$, then $e^{i\phi}$ is singular, for all other points on C_2 are interior to the unit circle. Conversely, if $e^{i\phi}$ is singular, we must have $R_1 = R_2 = 1 - b$. Hence the requirement $R_1 = 1 - b$ is necessary and sufficient in order that $e^{i\phi}$ be singular. Similarly, the necessary and sufficient condition that $e^{i\phi}$ be regular is $R_1 > 1 - b$.

Expressing R_1 in terms of the c_n, we have

(A) *If the following inequality holds*:

$$R_1 = \frac{1}{\overline{\lim}\sqrt[n]{|c_n|}} > 1 - b, \quad \overline{\lim}\sqrt[n]{|c_n|} < \frac{1}{1-b}, \tag{2}$$

then $e^{i\phi}$ *is regular*;

(B) *If*

$$R_1 = \frac{1}{\overline{\lim}\sqrt[n]{|c_n|}} = 1 - b, \quad \overline{\lim}\sqrt[n]{|c_n|} = \frac{1}{1-b}, \tag{3}$$

then $e^{i\phi}$ *is singular.*

If, in (*A*) and (*B*), the c_n are replaced by their values given by (1), we may say that the point $e^{i\phi}$ is singular if for an arbitrary $\epsilon > 0$ we have, for an infinity of values of m,

$$|c_m| = |a_m + (m+1)a_{m+1}be^{i\phi} + \cdots + C^q_{m+q}a_{m+q}b^q e^{qi\phi} + \cdots|$$

$$> \left(\frac{1-\epsilon}{1-b}\right)^m, \tag{5}$$

and $e^{i\phi}$ is regular if there exists an $\epsilon > 0$ so small that

$$|c_m| < \left(\frac{1-\epsilon}{1-b}\right)^m \tag{6}$$

for $m > m_0$.

In what follows we shall use the fact that *the left hand side of* (5) *and* (6) *may be replaced by a finite sum of terms* occurring in this expression, *e.g.*, by

$$| a_m + (m+1)a_{m+1}be^{i\phi} + \cdots + C^q_{m+q}a_{m+q}b^q e^{qi\phi} |,$$

where q is arbitrary, but $q \geq p = p(m)$, where this last number depends only on m. This principle, due to Hadamard,[1] has been employed by Fabry, Leau and others in obtaining important results.

54. We shall prove the existence of the number $p = p(m)$. Suppose for simplicity that $\phi = 0$. Since the quantity $C^q_{m+q}m^m q^q$ is a term of the expansion of $(m+q)^{m+q}$ we have

$$C^q_{m+q}m^m q^q < (m+q)^{m+q} \tag{7}$$

Moreover, since this term is the largest in the expansion,[2] and since there are $m+q+1$ terms, we have

$$C^q_{m+q}m^m q^q > \frac{(m+q)^{m+q}}{m+q+1}. \tag{8}$$

From (7) and (8),

$$\lim_{m\to\infty} \sqrt[m]{\frac{(m+q)^{m+q}}{m^m q^q} b^q} \geq \lim_{m\to\infty} \sqrt[m]{C^q_{m+q}b^q}$$

$$\geq \lim_{m\to\infty} \sqrt[m]{\frac{(m+q)^{m+q}}{(m+q+1)m^m q^q} b^q}, \tag{9}$$

where q may be supposed to vary with m, and where the existence of the limits is assumed.

[1] *loc. cit.*

[2] Write $(m+q)^{m+q} = \alpha_0 + \alpha_1 + \cdots + \alpha_{m+q}$, where

$$\alpha_k = \frac{(m+q)!}{(m+q-k)!\,k!} m^{m+q-k} q^k.$$

Then

$$\alpha_k = \frac{m+q-k+1}{k} \frac{q}{m} \alpha_{k-1},$$

so that if $k \leq q$, then $\alpha_k > \alpha_{k-1}$; whereas if $k > q$, $\alpha_k < \alpha_{k-1}$. The maximum term in the expansion of $(m+q)^{m+q}$ is therefore

$$\alpha_q = C^q_{m+q}\, m^m q^q. \qquad \text{[Editor.]}$$

From (7), and from the fact that the radius of convergence is 1 ($|a_{m+q}| < (1+\eta)^{m+q}$ for $\eta > 0$, arbitrary but fixed, m sufficiently large), we have

$$\sqrt[m]{C_{m+q}^{q} a_{m+q} b^{q}} < \left(1+\frac{q}{m}\right)^{\frac{m+q}{m}} m^{\frac{q}{m}} q^{-\frac{q}{m}} b^{\frac{q}{m}} (1+\eta)^{\frac{m+q}{m}}$$

$$= \left(1+\frac{q}{m}\right)^{1+\frac{q}{m}} \left(1+\eta\right)^{1+\frac{q}{m}} \left(\frac{bm}{q}\right)^{\frac{q}{m}} \tag{10}$$

$$= \left(1+\frac{q}{m}\right)\left(1+\eta\right)\left[\left(1+\eta\right) b \left(1+\frac{m}{q}\right)\right]^{\frac{q}{m}}.$$

Let l be a number satisfying the inequality

$$b(1+\eta) < l < 1.$$

To each integer m corresponds a number $p = p(m)$ such that for $q \geq p$ the following inequalities hold:

$$(1+\eta) b \left(1+\frac{m}{q}\right) < l, \tag{11}$$

$$\left(1+\frac{q}{m}\right)\left(1+\eta\right) l^{\frac{q}{m}} < \frac{l}{1-b}. \tag{12}$$

It is possible to choose b and l in such a way that (11) and (12) are surely satisfied for $\frac{p}{m} > \lambda'$, where λ' is an arbitrary fixed number greater than 1. In fact, on the one hand,

$$\lim_{b \to 0} (1+\eta) b \left(1+\frac{1}{\lambda'}\right) = 0;$$

the inequality

$$(1+\eta) b \left(1+\frac{1}{\lambda'}\right) < l \tag{11'}$$

is therefore satisfied for a pair of numbers b_1, l_1, both arbitrarily small, b_1 being chosen small enough with reference to l_1.

On the other hand,

$$\lim_{l\to 0}(1+\lambda')(1+\eta)l^{\lambda'-1}=0,$$

since $\lambda'>1$; and

$$\lim_{b\to 0}\frac{1}{1-b}=1.$$

Accordingly, for $l=l_1$, $b=b_1$, these numbers being sufficiently small, we have

$$(1+\lambda')(1+\eta)l^{\lambda'-1}<\frac{1}{1-b},$$

or

$$(1+\lambda')(1+\eta)l^{\lambda'}<\frac{l}{1-b}. \tag{12'}$$

Hence there exists a pair of numbers l_1, b_1, arbitrarily small, such that (11′) and (12′) are verified. We may take $l_1<\frac{1}{e}$.

Under the same conditions, the two quantities

$$(1+\eta)b_1\left(1+\frac{1}{x}\right),\quad (1+\eta)(1+x)l_1^{x-1},$$

$l_1\left(<\frac{1}{e}\right)$ and b_1 being fixed, decrease as x increases. The truth of this statement is obvious for the first expression; as for the second, its derivative with respect to x is negative when $l_1<\frac{1}{e}$. Since (11′) and (12′) hold for $l=l_1<\frac{1}{e}$, $b=b_1$, and $\lambda'>1$, it follows from what we have just seen that these inequalities are verified *a fortiori* if we replace λ' by $\frac{q}{m}>\lambda'$.

It is therefore always possible to satisfy (11) *and* (12) *by a suitable choice of* l *and* b, *and by letting* p *be arbitrary but greater than* $m\lambda'$, *where* $\lambda'>1$ *is fixed.* Hereafter it will be assumed that b, l and p have been so chosen.

From (11),

$$\left[(1+\eta)b\left(1+\frac{m}{q}\right)\right]^{\frac{q}{m}} < l^{\frac{q}{m}}, \quad q \geq p; \tag{13}$$

consequently, from (10) and (12),

$$|C_{m+q}^{q} a_{m+q} b^{q}| < \left(\frac{l}{1-b}\right)^{m}, \quad q \geq p. \tag{14}$$

Now, from the inequality

$$|a_{m+q}| < (1+\eta)^{m+q},$$

we have

$$\begin{aligned} &|C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + C_{m+q+2}^{q+2} a_{m+q+2} b^{q+2} + \cdots\cdot| \\ &< C_{m+q+1}^{q+1}(1+\eta)^{m+q+1} b^{q+1} + C_{m+q+2}^{q+2}(1+\eta)^{m+q+2} b^{q+2} \\ &\qquad + \cdots\cdot, \quad q \geq p. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{C_{m+r+1}^{r+1}(1+\eta)^{m+r+1} b^{r+1}}{C_{m+r}^{r}(1+\eta)^{m+r} b^{r}} &= b(1+\eta)^{\frac{m+r+1}{r+1}} \\ &< b\left(1+\frac{m}{r}\right)(1+\eta), \end{aligned} \tag{15}$$

and from (11),

$$\frac{C_{m+r+1}^{r+1}(1+\eta)^{m+r+1} b^{r+1}}{C_{m+r}^{r}(1+\eta)^{m+r} b^{r}} < l. \tag{16}$$

Combining this inequality with (14), we have

$$\begin{aligned} &|C_{m+q}^{q} a_{m+q} b^{q} + C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + \cdots\cdot| \\ &< |C_{m+q}^{q} a_{m+q} b^{q}|(1 + l + l^{2} + \cdots\cdot) \\ &< \left(\frac{l}{1-b}\right)^{m} \frac{1}{1-l}, \end{aligned}$$

and therefore

$$\begin{aligned} \overline{\lim_{m\to\infty}} \sqrt[m]{|C_{m+q}^{q} a_{m+q} b^{q} + C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + \cdots\cdot} \\ < \frac{1}{1-b} \quad \text{if} \quad q \geq p. \end{aligned}$$

But if $\{A_n\}$, $\{B_n\}$ are two sequences such that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n|} = A,$$

and

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|B_n|} = B < A,$$

then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n + B_n|} = A, \tag{18}$$

which is equivalent to the statement that if the radius of convergence of $\Sigma B_n x^n$ is greater than that of $\Sigma A_n x^n$, then the series $\Sigma(A_n + B_n)x^n$ has the same radius of convergence as $\Sigma A_n x^n$.

Let

$$\left.\begin{aligned} A_m &= a_m + (m+1)a_{m+1}b + \cdots + C_{m+q}^{q} a_{m+q} b^q, \\ B_m &= C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + \cdots. \end{aligned}\right\} \tag{19}$$

Then the point 1 is singular if

$$\sqrt[n]{|a_m + C_{m+1}^{1} a_{m+1} b + \cdots + C_{m+q}^{q} a_{m+q} b^q|} = \sqrt[n]{|A_m + B_m|} > \frac{1-\epsilon}{1-b}$$

for an infinity of m, and for every $\epsilon > 0$. The point 1 is regular if for an arbitrary ϵ,

$$\sqrt[m]{|a_m + C_{m+1}^{1} a_{m+1} b + \cdots + C_{m+q}^{q} a_{m+q} b^q|} < \frac{1-\epsilon}{1-b}$$

for m sufficiently large, and for b sufficiently small, depending on ϵ.

In general, the point $e^{i\phi}$ is regular if

$$\overline{\lim_{m\to\infty}} \sqrt[m]{|a_m + C_{m+1}^{1} a_{m+1} b e^{i\phi} + \cdots + C_{m+q}^{q} a_{m+q} b^q e^{qi\phi}|} < \frac{1}{1-b},$$

and singular if

$$\overline{\lim_{m\to\infty}}\sqrt[m]{|a_m + C_{m+1}^1 a_{m+1} b e^{i\phi} + \cdots + C_{m+q}^q a_{m+q} b^q e^{qi\phi}|} = \frac{1}{1-b}, \tag{5'}$$

for $q \geq p$, and for b sufficiently small.

55. THEOREM 1: *The circle of convergence of the series*

$$\sum a_{\lambda_n} x^{\lambda_n} = \sum k_m x^m, \tag{20}$$

where

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_n} > \lambda > 1, \quad n = 1, 2, \cdots, \tag{20'}$$

λ *fixed, is a cut.*[1]

We assume that the radius of convergence is 1. Corresponding to each m, choose a number q' such that

$$\lim_{m\to\infty} \frac{q'}{m} = \frac{b}{1-b}, \tag{21}$$

where b is given as in the previous section. Then

$$\begin{aligned} &\lim_{m\to\infty} \sqrt[m]{\frac{(m+q')^{m+q'}}{m^m q'^{q'}} b^{q'}} \\ &= \lim_{m\to\infty} \left(1 + \frac{q'}{m}\right)^{1+\frac{q'}{m}} m^{\frac{q'}{m}} q'^{-\frac{q'}{m}} b^{\frac{q'}{m}} \\ &= \left(\frac{1}{1-b}\right)^{\frac{1}{1-b}} \left(\frac{1}{1-b}\right)^{\frac{-b}{1-b}} = \frac{1}{1-b}. \end{aligned} \tag{22}$$

On the other hand,

$$\lim_{m\to\infty} \sqrt[m]{m + q' + 1} = 1.$$

Hence, from (22),

$$\lim_{m\to\infty} \sqrt[m]{\frac{(m+q')^{m+q'}}{(m+q'+1)m^m q'^{q'}} b^{q'}} = \frac{1}{1-b},$$

[1] See footnote, p. 351

and from (9),

$$\lim_{m\to\infty} \sqrt[m]{C^{q'}_{m+q'} b^{q'}} = \frac{1}{1-b}. \tag{23}$$

We choose b so small that

$$\frac{b}{1-b} < 1. \tag{24}$$

Let λ' be a number satisfying the inequality

$$1 < \lambda' < \lambda, \tag{25}$$

and define the partial sequence $\{m_n\}$ by

$$m_n = E\,[\lambda_n(1-b)].^1 \tag{26}$$

We have

$$\lim_{n\to\infty} \frac{\lambda_n - m_n}{m_n} = \frac{b}{1-b}. \tag{26'}$$

In (5') let m take on the values m_n, placing

$$q = \lambda_{n+1} - m_n - 1. \tag{26''}$$

Since, from (20'), we have

$$\lambda_{n-1} < \frac{\lambda_n}{1+\lambda},$$

where $\lambda > \frac{b}{1-b}$, it follows that $\lambda_{n-1} < \lambda_n(1-b)$, hence, from (26),

$$m_n > \lambda_{n-1}.$$

Noting that $\lambda_n > m_n$, we obtain from (26''),

$$\frac{q}{m_n} = \frac{\lambda_{n+1}}{m_n} - 1 - \frac{1}{m_n} > \frac{\lambda_{n+1}}{\lambda_n} - 1 - \frac{1}{m_n}.$$

Hence, for n sufficiently large,

$$\frac{q}{m_n} > \lambda',$$

[1] $E[x]$ denotes the smallest integer $\geq x$.

since $\lim_{n\to\infty} \frac{1}{m_n} = 0$. But we have seen (page 343) that, under these conditions, the inequalities (11) and (12) hold, provided further that l and b are suitably chosen, b sufficiently small with reference to l, which is arbitrarily small. If then q is chosen so as to satisfy (26″), *i.e.*, if

$$m_n + q = \lambda_{n+1} - 1, \tag{27}$$

then q may be substituted in (5′) and (6′) in order to obtain the criterion for determining whether or not a given point on the circle of convergence is singular. From (27), all the coefficients k_m from k_{m_n} to k_{m_n+q} vanish except k_{λ_n}, hence

$$\begin{aligned} k_{m_n} + C^1_{m+1} k_{m_n+1} b e^{i\phi} + \cdots + C^q_{m_n+q} k_{m_n+q} b^q e^{qi\phi} \\ = C^{\lambda_n - m_n}_{\lambda_n} k_{\lambda_n} b^{\lambda_n - m_n} e^{(\lambda_n - m_n)i\phi} \\ = C^{q'}_{\lambda_n} k_{\lambda_n} b^{q'} e^{q'i\phi}, \end{aligned} \tag{28}$$

where $q' = \lambda_n - m_n$. By (26), the number q' satisfies the requirement (21) if m takes on the values m_n.

From (23),

$$\lim_{n\to\infty} \sqrt[m_n]{| C^{q'}_{\lambda_n} b^{q'} e^{q'i\phi} |} = \frac{1}{1-b}.$$

Now there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that

$$\lim_{i\to\infty} \sqrt[\lambda_{n_i}]{| k_{\lambda_{n_i}} |} = 1.$$

But from (26),

$$\lim_{i\to\infty} \frac{\lambda_{n_i}}{m_{n_i}} = \frac{1}{1-b},$$

and therefore

$$\lim_{i\to\infty} \sqrt[m_{n_i}]{| k_{\lambda_{n_i}} |} = \left[\lim_{i\to\infty} \sqrt[\lambda_{n_i}]{| k_{\lambda_{n_i}} |}\right]^{\frac{\lambda_{n_i}}{m_{n_i}}} = 1.$$

Hence

$$\overline{\lim_{n\to\infty}} \sqrt[m_n]{|k_{\lambda_n}|} = 1. \tag{30}$$

We obtain from (29) and (30),

$$\overline{\lim_{n\to\infty}} \sqrt[m_n]{|C^{q'}_{\lambda_n} b^{q'} e^{q'i\phi} k_{\lambda_n}|} = \frac{1}{1-b},$$

and from (28),

$$\overline{\lim_{n\to\infty}} \sqrt[m_n]{|k_{m_n} + C^1_{m_n+1} k_{m_n+1} b e^{i\phi} + \cdots + C^q_{m_n+q} k_{m_n+q} b^q e^{qi\phi}|} = \frac{1}{1-b}.$$

Comparison with (5′) shows that every point $e^{i\phi}$ is singular. Theorem 1 is therefore proved.

56. Theorem 2: *The circle of convergence of the series*

$$\Sigma a_{\lambda_n} x^{\lambda_n}$$

is a cut provided that there exists a number $\sigma < 1$ *such that the series*

$$\Sigma \frac{1}{\lambda_n^\sigma}$$

converges.

Let $\Sigma a_{\lambda_n} x^{\lambda_n} = \Sigma k'_m x^m$. From the sequence $\{\lambda_n\}$ we may extract a subsequence $\{\mu_m\}$ satisfying the conditions

$$\frac{\mu_{m+1} - \mu_m}{\mu_m} > \lambda > 1, \tag{31}$$

λ being fixed; and

$$\lim_{m\to\infty} \sqrt[\mu_m]{|a_{\mu_m}|} = \lim_{m\to\infty} \sqrt[\mu_m]{|k'_{\mu_m}|} = 1. \tag{32}$$

Denote by $\{\eta_k\}$ the sequence of integers contained in $\{\lambda_n\}$ and having no element in common with $\{\mu_m\}$. The sequence $\{\eta_k\}$ clearly has the property that the series $\Sigma \frac{1}{\eta_k^\sigma}$ converges.

We may form an integral function

$$g(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{\eta_k}\right)$$

having the points η_k and no others as zeros. By Theorem 5, Chapter V, the function defined by $\Sigma g(n)x^n$ has just one singularity in the entire plane.

On the other hand, it follows from the work of Poincaré [1] that the function $g(z)$ has the property that

$$|g(z)| > e^{-r^{\sigma+\epsilon}}, \quad r = |z|, \quad \epsilon > 0,$$

for $r > r_\epsilon$, the point z being such that within the ring formed by the circles with centers at the origin, radii $|z| - 1$, $|z| + 1$, respectively, the function $g(z)$ does not vanish.

But the number μ_m has the property required for z, from the way in which $g(z)$ was formed; hence, for k sufficiently large,

$$|g(\mu_k)| > e^{-\mu_k^{\sigma+\epsilon}}, \tag{33}$$

for $\epsilon > 0$ arbitrary.

Now the series

$$\Sigma g(n)k_n'x^n = \Sigma d_n x^n \tag{34}$$

has only the d_{μ_m} as non-zero coefficients, since for the indices n different from μ_m we have either $g(n) = 0$ or $k_n' = 0$. Hence we may write

$$\Sigma g(n)k_n'x^n = \Sigma d_{\mu_m}x^{\mu_m}.$$

But we have, from (33),

$$\overline{\lim_{m\to\infty}} \sqrt[\mu_m]{|g(\mu_m)|} \geq \overline{\lim_{m\to\infty}}\, e^{-\mu_m^{\sigma+\epsilon-1}}$$

$$= 1, \text{ since } 0 < \sigma + \epsilon < 1.$$

[1] Bull. de la Soc. Math. de France, t. 11 (1883), p. 136.

Since $\Sigma g(n)x^n$ converges within the unit circle, we have

$$\overline{\lim_{m\to\infty}} \sqrt[\mu_m]{|g(\mu_m)|} \leq 1.$$

Hence

$$\overline{\lim_{m\to\infty}} \sqrt[\mu_m]{|g(\mu_m)|} = 1.$$

Hence, from (32) and (34),

$$\overline{\lim_{m\to\infty}} \sqrt[\mu_m]{|d_{\mu_m}|} = 1,$$

that is, the series $\Sigma d_{\mu_m} x^{\mu_m}$ has its radius of convergence equal to 1. The function defined by this series accordingly has the unit circle as a cut, since the sequence $\{\mu_m\}$ satisfies (31).

Now

$$\sum d_{\mu_m} x^{\mu_m} = \sum d_n x^n = \sum g(n) k_n x^n,$$

and since $\Sigma g(n)x^n$ defines a function having the point 1 as its only singularity, it follows from Hadamard's theorem on the multiplication of singularities that $\Sigma k_n x^n$ likewise has its circle of convergence as a cut. For otherwise the circle of convergence of $\Sigma d_n x^n$ would not be a cut. The proof is therefore complete.

57. Theorem 3: *The circle of convergence of the series* $\Sigma a_{\lambda_n} x^{\lambda_n}$ *is a cut provided that*

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_n^{\alpha}} > h,$$

where α, h *are arbitrary but fixed positive numbers.*[1]

[1] When $\alpha = 1$, the theorem reduces to Theorem 1, Ch. V. Taking $\alpha = \frac{1}{2}$, we have the theorem of Borel, Jour. de Math., t. ii (1896), p. 441. Finally, the most general theorem, viz., that the circle of convergence is a cut if $\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \infty$, is due to Fabry, Ann. de l'École Norm. Sup., t. xiii (1896), p. 367.

We have, by hypothesis,

$$\lambda_{n+p} - \lambda_{n+p-1} > h\lambda^{\alpha}_{n+p-1}.$$

Obviously $\lambda_{n+k} > k+1$ for k sufficiently large. We may therefore assume that this inequality holds for all k. We have, then,

$$\lambda_{n+p} - \lambda_{n+p-1} > hp^{\alpha}.$$

Hence,

$$\lambda_{n+p} - \lambda_n > h(1^{\alpha} + 2^{\alpha} + \cdots + p^{\alpha}),$$

$$\lambda_{n+p} > h(1^{\alpha} + 2^{\alpha} + \cdots + p^{\alpha})$$

$$> h\int_0^p x^{\alpha} dx$$

$$= \frac{h}{\alpha+1} p^{\alpha+1}$$

We may choose a number r, $0 < r < 1$, such that

$$r(\alpha+1) > 1 + \delta, \quad \delta > 0,$$

provided δ is small enough. Consequently

$$\lambda^r_{n+p} > \left(\frac{h}{\alpha+1}\right)^r p^{r(\alpha+1)}$$

$$> \left(\frac{h}{\alpha+1}\right)^r p^{1+\delta}$$

The series $\sum \frac{1}{\lambda_n^r}$ is therefore convergent. Hence, by Theorem 2, the circle of convergence of the given series is a cut.

This transition from Theorem 1 to Theorem 3 by means of Theorem 2 is due to Faber.[1] Starting with Theorem 1 he also proves by an analogous process the theorem of Fabry referred to in the footnote on p. 351. The latter theorem includes as particular cases the theorems of Hadamard and Borel.

S. MANDELBROJT.

[1] Sitzungsb. der Math.-Phys. Classe der Münicher Acad. (1904), p. 62.

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